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A stochastic approach to stability in stochastic programming

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Abstract

A random optimization problem

$$P_0 \qquad \min_{x \in \Gamma_0(\omega)} f_0(x, \omega), \quad \omega \in \Omega,$$

is approximated by a sequence of random surrogate problems $(P_n)_{n \in \mathbb{N}}$ with

$$P_n \qquad \min_{x \in \Gamma_n(\omega)} f_n(x, \omega), \quad \omega \in \Omega$$

($[\Omega, \Sigma, P]$ a given probability space).

We investigate the convergence almost surely and in probability of the optimal values and the solution sets. The results can be regarded as random versions of well-known stability statements of parametric programming. Semicontinuous convergence (almost surely, in probability) of sequences of random functions is a crucial assumption in this framework and will be investigated in more detail.

Keywords: Stochastic programming; Stability; Convergence almost surely; Convergence in probability

1. Introduction

Often mathematical programming problems are not completely known. Mostly certain quantities have to be approximated by estimates and the decision maker solves a surrogate problem instead of the true one. He hopes that good estimates will provide optimal values and solution sets which are "close" to the true optimal value and the true solution set, respectively.

But not in every case this hope is justified, thus there is the need for statements on – at least – the convergence of the optimal values and the solution sets of the approximate problems to the true quantities.

There are several approaches to derive such results. Especially stability theory of (deterministic) parametric programming (cf. [1, 17]), proved as a tool which may be successfully employed in the stochastic programming case too, for instance regarding the probability measures as parameters

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[6, 11, 18, 21–23, 33]. Particularly for the stability analysis of solution procedures, where the true probability measure is approximated by simpler (deterministic) ones, this approach is very useful.

However, the random nature of the surrogate problems under consideration raises the question for approaches which are directly adapted to the randomness, taking into account stochastic convergence notions. Thus [7, 13, 15] deal with convergence almost surely, [14, 29, 31] investigate convergence in probability, while [26, 32] consider convergence in distribution. Reference [20] investigates convergence of measurable selections almost surely, in probability and in mean and applies the results to stochastic programming.

In general, convergence in a random sense being weaker than convergence in the deterministic sense, it can be proved for a wider class of problems.

The present paper will provide a unifying framework for stability considerations in terms of convergence almost surely and convergence in probability. Convergence in distribution requires special preliminaries and will be dealt with elsewhere. We assume that the objective function and the constraints are approximated simultaneously and we try to present the results in a form that allows the direct application of limit theorems in probability theory and asymptotic results in statistics.

Since the derivation of many estimators is based on the minimization (with or without constraints) of a certain function, the present approach may also be used to prove consistency statements for estimators (cf. [7, 10]). Consistency in turn is often among the assumptions in theorems on the asymptotic distribution of solutions to stochastic programming problems [28].

Concerning the convergence notions for random sets and random functions we essentially rely on the papers by Salinetti and Wets [25, 26]. However, in general we do not assume that the multifunctions and epigraphs of functions under consideration are closed-valued, because we are aiming at results which are stochastic analogues of well-known stability theorems of parametric programming. For the same reason we introduce semi-(continuous) convergence concepts for random sets, random functions and random variables. Salinetti and Wets dealt with random original problems. We shall prove the stability statements in the general framework of [26], but we shall also point out that in the special case of a deterministic original problem there are simpler sufficient conditions for the semicontinuous convergence of random functions.

The paper is organized as follows: In Section 2 we introduce and discuss the semiconvergence notions for random sets and random functions. Section 3 contains the main results on the approximation of the constraint set, Section 4 deals with the convergence of the optimal values and the solution sets. In Section 5 we investigate a deterministic original problem.

Let $[\Omega, \Sigma, P]$ be a given complete probability space and consider the following original problem:

$$P_0 \qquad \min_{x \in \Gamma_0(\omega)} f_0(x, \omega), \quad \omega \in \Omega.$$

We assume that P_0 is approximated by a sequence of surrogate problems

$$P_n \qquad \min_{x \in \Gamma_n(\omega)} f_n(x, \omega), \quad \omega \in \Omega, \ n \in \mathbb{N}.$$

Γ_n, $n \in \mathbb{N} \cup \{0\}$, are multifunctions which map into the power set of $\mathbb{R}^p$. We shall not suppose that the Γ_n are closed-valued, but it is assumed throughout the paper that the graph of the Γ_n is an element of the σ-field $\Sigma \otimes \Sigma^p$, which is generated by Σ and the Borel σ-field Σ^p of $\mathbb{R}^p$. This condition implies that

$$\Gamma_n^{-1}(B) = \{\omega \in \Omega\colon \Gamma_n(\omega) \cap B \neq \emptyset\} \in \Sigma \quad \text{for all } B \in \Sigma^p$$

(cf. [9]), i.e., Γ_n is a measurable multifunction.

The functions f_n take values in $(-\infty, +\infty]$, they are supposed to be $(\Sigma^p \otimes \Sigma, \bar{\Sigma}^1)$-measurable, where $\bar{\Sigma}^1$ denotes the σ-field of Borel sets of the extended reals $\bar{\mathbb{R}}^1 = \mathbb{R}^1 \cup \{-\infty\} \cup \{+\infty\}$. The constraint set being often specified by inequality constraints, we shall assume that the multifunctions $\Gamma_n, n \in \mathbb{N} \cup \{0\}$, are given in the form

$$\Gamma_n(\omega) = Q_n(\omega) \cap \tilde{\Gamma}_n(\omega),$$

where $Q_n | \Omega \to 2^{\mathbb{R}^p}$ is a multifunction with measurable graph and

$$\tilde{\Gamma}_n(\omega) := \{x \in \mathbb{R}^p\colon g_n^j(x, \omega) \leqslant 0, j \in J\}.$$

The functions g_n^j, $n \in \mathbb{N} \cup \{0\}, j \in J$, have to be $(\Sigma^p \otimes \Sigma, \bar{\Sigma}^1)$-measurable, J is a countable index set.

Under the given conditions we have Graph $\Gamma_n \in \Sigma \otimes \Sigma^p$. By Φ_n we denote the optimal value and by Ψ_n the solution set ($n \in \mathbb{N} \cup \{0\}$):

$$\Phi_n(\omega) := \begin{cases} \inf\limits_{x \in \Gamma_n(\omega)} f_n(x, \omega) & \text{if } \Gamma_n(\omega) \neq \emptyset, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\Psi_n(\omega) := \{x \in \Gamma_n(\omega)\colon f_n(x, \omega) = \Phi_n(\omega)\}.$$

Especially, if $\Phi_n(\omega) = +\infty$ we obtain either $\Gamma_n(\omega) = \emptyset$, hence $\Psi_n(\omega) = \emptyset$, or $f_n(x, \omega) = +\infty$ $\forall x \in \Gamma_n(\omega)$, hence $\Psi_n(\omega) = \Gamma_n(\omega)$.

Φ_n is measurable according to [3, Lemma III.39]. Consequently functions $f_{\Phi, n} | \mathbb{R}^p \times \Omega \to \mathbb{R}^1$, $n \in \mathbb{N}$, defined by

$$f_{\Phi, n}(x, \omega) := \begin{cases} f_n(x, \omega) - \Phi_n(\omega) & \text{if } \Phi_n(\omega) < \infty, \\ -\infty & \text{if } \Phi_n(\omega) = \infty, \end{cases}$$

are $(\Sigma^p \otimes \Sigma, \bar{\Sigma}^1)$-measurable. Because of $\Psi_n(\omega) = \Gamma_n(\omega) \cap \{x \in \mathbb{R}^p\colon f_{\Phi, n}(\omega) \leqslant 0\}$, we obtain

$$\text{Graph } \Psi_n = \text{Graph } \Gamma_n \cap \{(x, \omega)\colon f_{\Phi, n}(x, \omega) \leqslant 0\},$$

i.e., the desired measurability assumptions are fulfilled. Note that we have to require stronger measurability conditions than in [25], because we do not assume that the multifunctions under consideration are closed-valued.

The consideration of random original problems offers the possibility to investigate the optimization of random processes, however, as already mentioned in the introduction, in this paper we are especially interested in deterministic original problems P_0, where $f_0(x, \omega) \equiv f_{0,D}(x)$ and $\Gamma_0(\omega) \equiv \Gamma_{0,D}$ $\forall \omega \in \Omega$. Various problems may be described in this form, see the following examples.

If $f_{0,D}$ depends on an unknown parameter $\lambda_0 \in \mathbb{R}^m$, $f_{0,D}(x) = \hat{f}(x, \lambda_0)$, then, replacing λ_0 by an estimate $\Lambda_n | [\Omega, \Sigma, P] \to [\mathbb{R}^m, \Sigma^m]$, we obtain surrogate objective functions $f_n(x, \omega) := \hat{f}(x, \Lambda_n(\omega))$ with the desired measurability property, provided that $\hat{f}$ is $(\Sigma^p \otimes \Sigma^m, \bar{\Sigma}^1)$-measurable. In the same way one gets random surrogate functions for the functions $g^j_{0,D}$, describing the inequality constraints.

An important application of our results will be the investigation of stochastic programming problems. Suppose that a decision maker has to choose a decision x, but he knows that the reward he will obtain depends on the realization of a random variable Z which he will observe in the future. Additionally, there are several restrictions that have to be taken into account, which may depend on Z, too. If there is the possibility to compensate the violation of the restrictions by a second action (which will cause additional costs), the problem can be formulated as a so-called recourse or two-stage problem. Then, in general, one tries to optimize the expected total reward. Such a model fits into our framework with

$$f_{0,D}(x) := f_{0,1}(x) + \mathrm{E}H(x,Z),$$

where E denotes the expectation. $H(x,z)$ is the (minimal) cost for the compensation, it depends on the chosen decision x and the observed realization of Z. If compensation is impossible, i.e., the constraint set $\Gamma_{0,2}(x,z) = \emptyset$, we put $H(x,z) = \infty$. There may be restrictions on x that do not depend on Z, they describe the constraint set $\Gamma_{0,D}$. Two-stage problems have been investigated in detail, concerning their stability properties, see [12, 15, 18, 22].

Sometimes it is difficult or even impossible to set costs for the violation of the restrictions, for instance, if the violation of the restrictions could cause a catastrophe. But often one does not find any decision which satisfies the restrictions for all possible realizations of Z, hence one usually imposes the condition that the probability for the violation of important restrictions does not exceed a given small probability level. The expected reward is commonly used as objective function.

We shall write the objective function and the constraint set of these so-called chance-constrained (or probabilistic-constrained) problems in the following form:

$$f_{0,D}(x) := \mathrm{E}\varphi(x,Z),$$

$$\Gamma_{0,D} = \{x \in \mathbb{R}^p \colon P\{\omega \colon \gamma^j_l(x, Z(\omega)) \leqslant 0,\ l = 1, \dots, q_j\} \geqslant \eta^j, j \in J\}.$$

The functions $\varphi : \mathbb{R}^p \times \mathbb{R}^m \to \bar{\mathbb{R}}^1$ and $\gamma^j_l : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^1$ are supposed to be measurable with respect to the second variable, and we assume that $\mathrm{E}\varphi(x,Z)$ exists for all $x \in \Gamma_{0,D}$. We allow for two kinds of indices at γ in order to cover joint and individual probabilistic constraints. The values $\eta^j \in (0,1)$ are given probability levels.

$\Gamma_{0,D}$ may be written in the following form:

$$\Gamma_{0,D} = \{x \in \mathbb{R}^p \colon \eta^j - \mathrm{E}\chi_{M^j(x)}(Z) \leqslant 0,\ j \in J\},$$

where

$$M^j(x) = \{z \in \mathbb{R}^m \colon \gamma^j_l(x,z) \leqslant 0,\ l = 1, \dots, q^j\},$$

and for $A \in \Sigma^m$,

$$\chi_A(z) = \begin{cases} 1 & \text{if } z \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Stability of chance-constrained problems was considered in [13, 21, 23, 24, 31–33].

It is evident that the two-stage problems and the chance-constrained problems heavily depend on the distribution of Z. On the other hand, only in a few cases this distribution is completely known. Therefore stability with respect to the distribution of the underlying random variable plays a crucial role in stochastic programming.

If the distribution of Z is known up to a certain parameter, say λ_0, which has to be estimated, one is in the framework described above.

Further, estimating the distribution function of Z by the empirical distribution function, we obtain for $f_{0,D}(x) = \mathrm{E}\varphi(x,Z)$ the surrogate function

$$f_n(x,\omega) = \frac{1}{n}\sum_{i=1}^{n} \varphi(x, Z_i(\omega)),$$

and for $g^j_{0,D}(x) := \eta^j - \mathrm{E}\chi_{M^j(x)}(Z)$ the surrogate function

$$g^j_n(x,\omega) = \eta^j - \frac{1}{n}\sum_{i=1}^{n} \chi_{M^j(x)}(Z_i(\omega)), \quad Z_i \text{ i.i.d.}$$

For two-stage problems one can proceed in a similar way.

If Z has a density function π_0, one may employ density estimators, for instance kernel estimators. Then the objective function

$$f_{0,D}(x) = \int_{\mathbb{R}^m} \varphi(x,z)\pi_0(z)\,\mathrm{d}z$$

leads to

$$f_n(x,\omega) = \int_{\mathbb{R}^m} \varphi(x,z)\pi_n(z,\omega)\,\mathrm{d}z,$$

where

$$\pi_n(z,\omega) = \frac{1}{n(h_n)^m}\sum_{i=1}^{n} k\left(\frac{z - Z_i(\omega)}{h_n}\right).$$

$(h_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers tending to zero, and the kernel k is supposed to be a nonnegative Borel function with $\int_{\mathbb{R}^m} k(z)\,\mathrm{d}z = 1$.

2. Convergence almost surely and in probability

Let $\{G_n, n \in \mathbb{N}\cup\{0\}\}$ be a family of multifunctions $G_n : \Omega \to 2^{\mathbb{R}^p}$ with measurable graphs.

Definition 2.1. The sequence $(G_n)_{n\in\mathbb{N}}$ is said to be

(i) *upper semiconvergent almost surely to* G_0 $\left(G_n \xrightarrow{\text{u-a.s.}} G_0\right)$ if

$$\limsup_{n\to\infty} G_n(\omega) \subset G_0(\omega) \quad \text{P-a.e.},$$

(ii) *lower semiconvergent almost surely to* G_0 $\left(G_n \xrightarrow{\text{l-a.s.}} G_0\right)$ if

$$\liminf_{n\to\infty} G_n(\omega) \supset G_0(\omega) \quad \text{P-a.e.}$$

The "$\liminf_{n\to\infty}$" and "$\limsup_{n\to\infty}$" in this definition and throughout the paper is understood in the Kuratowski and Mosco-sense:

$$\limsup_{n\to\infty} S_n := \left\{x \in \mathbb{R}^p\colon \exists (x_{n_k})_{k\in\mathbb{N}} \text{ with } \lim_{k\to\infty} x_{n_k} = x \text{ and } x_{n_k} \in S_{n_k}\ \forall k \in \mathbb{N}\right\},$$

$$\liminf_{n\to\infty} S_n := \left\{x \in \mathbb{R}^p\colon \exists (x_n)_{n\in\mathbb{N}} \text{ with } \lim_{n\to\infty} x_n = x \text{ and } x_n \in S_n\ \forall n \geqslant n_0\right\}$$

($(S_n)_{n\in\mathbb{N}}$ denotes a sequence of subsets of $\mathbb{R}^p$).

Note that a sequence $(G_n)_{n\in\mathbb{N}}$ is convergent almost surely to G_0 $\left(G_n \xrightarrow{\text{a.s.}} G_0\right)$ according to the definition used in [25] if and only if

$$\left(G_n \xrightarrow{\text{u-a.s.}} G_0\right) \wedge \left(G_n \xrightarrow{\text{l-a.s.}} G_0\right).$$

We shall also use the designation upper and lower semiconvergence for sequences $(S_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^p$ and $S_0 \subset \mathbb{R}^p$.

Salinetti and Wets [25] introduced convergence in probability for sequences of closed-valued measurable multifunctions. They essentially relied on the following equivalence for families $\{S_n, n \in \mathbb{N} \cup \{0\}\}$ of closed sets:

$$\left(\limsup_{n\to\infty} S_n \subset S_0\right) \Leftrightarrow \left(\forall \varepsilon > 0\colon \lim_{n\to\infty} (S_n \setminus U_\varepsilon S_0) = \emptyset\right),$$

$$\left(\liminf_{n\to\infty} S_n \supset S_0\right) \Leftrightarrow \left(\forall \varepsilon > 0\colon \lim_{n\to\infty} (S_0 \setminus U_\varepsilon S_n) = \emptyset\right).$$

($U_\varepsilon S$ denotes an ε-neighbourhood of $S \subset \mathbb{R}^p$:

$$U_\varepsilon S := \{x \in \mathbb{R}^p\colon \inf_{y\in S} d(x, y) < \varepsilon\},$$

where d is the Euclidean distance in $\mathbb{R}^p$.)

Unfortunately, if S_0 is not closed, the implication

$$\left(\forall \varepsilon > 0\colon \lim_{n\to\infty} (S_n \setminus U_\varepsilon S_0) = \emptyset\right) \Rightarrow \left(\limsup_{n\to\infty} S_n \subset S_0\right)$$

does not remain true, as the example

$$S_n = \left\{\frac{1}{n}\right\}, \quad n \in \mathbb{N}, \qquad S_0 = (0, 1)$$

shows. Therefore, allowing for multifunctions that are not closed-valued, we shall not obtain relations between (semi)convergence almost surely and in probability that are quite analogous to those known for random variables. True analogy holds for closed-valued measurable multifunctions only.

Let C^p denote the family of compact subsets of $\mathbb{R}^p$.

Definition 2.2. The sequence $(G_n)_{n\in\mathbb{N}}$ is said to be

(i) *upper semiconvergent in probability to* G_0 $\left(G_n \xrightarrow{\text{u-prob}} G_0\right)$ if

$$\forall \varepsilon > 0 \ \forall K \in C^p\colon \ \lim_{n\to\infty} P\{\omega\colon [G_n(\omega)\setminus U_\varepsilon G_0(\omega)]\cap K \neq \emptyset\} = 0,$$

(ii) *lower semiconvergent in probability to* G_0 $\left(G_n \xrightarrow{\text{l-prob}} G_0\right)$ if

$$\forall \varepsilon > 0 \ \forall K \in C^p\colon \ \lim_{n\to\infty} P\{\omega\colon [G_0(\omega)\setminus U_\varepsilon G_n(\omega)]\cap K \neq \emptyset\} = 0.$$

The sequence $(G_n)_{n\in\mathbb{N}}$ is convergent in probability to G_0 $\left(G_n \xrightarrow{\text{prob}} G_0\right)$ according to the definition in [25] if and only if

$$\left(G_n \xrightarrow{\text{u-prob}} G_0\right) \wedge \left(G_n \xrightarrow{\text{l-prob}} G_0\right).$$

The convergence in probability may be supplemented by a convergence rate, see [30, 31].

Salinetti and Wets [25] proved that convergence almost surely of closed-valued measurable multifunctions implies convergence in probability. A corresponding statement is valid for semiconvergence:

$$\left(G_n \xrightarrow{\text{l-a.s.}} G_0\right) \Rightarrow \left(G_n \xrightarrow{\text{l-prob}} G_0\right)$$

and

$$\left(G_n \xrightarrow{\text{u-a.s.}} G_0\right) \Rightarrow \left(G_n \xrightarrow{\text{u-prob}} G_0\right).$$

This statement can be proved by making use of the relations

$$\left(\limsup_{n\to\infty} G_n(\omega) \subset G_0(\omega)\right) \Rightarrow \left(\forall \varepsilon > 0\colon \lim_{n\to\infty} (G_n(\omega)\setminus U_\varepsilon G_0(\omega)) = \emptyset\right), \tag{2.1}$$

$$\left(\liminf_{n\to\infty} G_n(\omega) \supset G_0(\omega)\right) \Leftrightarrow \left(\forall \varepsilon > 0\colon \lim_{n\to\infty} (G_0(\omega)\setminus U_\varepsilon G_n(\omega)) = \emptyset\right) \tag{2.2}$$

and

$$\left(\lim_{n\to\infty} \tilde{G}_n(\omega) = \emptyset\right) \Leftrightarrow (\forall K \in C^p, \exists n_0(\omega), \forall n \geqslant n_0(\omega): \tilde{G}_n(\omega) \cap K = \emptyset), \tag{2.3}$$

with

$$\tilde{G}_n(\omega) = G_n(\omega) \setminus U_\varepsilon G_0(\omega) \quad \text{or} \quad \tilde{G}_n(\omega) = G_0(\omega) \setminus U_\varepsilon G_n(\omega).$$

If $G_0(\omega)$ is closed, one has an equivalence in condition (2.1), too.

The semiconvergence of $(S_n)_{n\in\mathbb{N}}$ can be regarded as semicontinuity of a multifunction $\hat{S}$ with $\hat{S}(n) := S_n$, $\hat{S}(\infty) := S_0$ at ∞. We prefer the notation semiconvergence instead of semicontinuity, because in our framework sequences seem to be the more natural way of description.

Lower semiconvergence of $(S_n)_{n\in\mathbb{N}}$ is essentially the same as lower semicontinuity of the multifunction $\hat{S}$ in the sense of Berge. Upper semicontinuity of $(S_n)_{n\in\mathbb{N}}$ corresponds to closedness of $\hat{S}$ if the "limit multifunction" G_0 is closed-valued. For details see [30].

The following proposition will be used in the proofs to the main results of this paper.

Proposition 2.3.

(i) $\left(G_n \xrightarrow{u\text{-}prob} G_0\right)$

$$\Rightarrow (\forall K \in C^p\ \exists (\alpha_n)_{n\in\mathbb{N}} \textit{ with } \alpha_n \downarrow 0:$$

$$\lim_{n\to\infty} P\{\omega: (G_n(\omega) \setminus U_{\alpha_n} G_0(\omega)) \cap K \neq \emptyset\} = 0).$$

(ii) $\left(G_n \xrightarrow{l\text{-}prob} G_0\right)$

$$\Rightarrow (\forall K \in C^p\ \exists (\alpha_n)_{n\in\mathbb{N}} \textit{ with } \alpha_n \downarrow 0:$$

$$\lim_{n\to\infty} P\{\omega: (G_0(\omega) \setminus U_{\alpha_n} G_n(\omega)) \cap K \neq \emptyset\} = 0).$$

Proof. (i) Let $K \in C^p$ be fixed and consider the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n = 1/(n+1)$. To ε_i there is an n_i such that

$$P\{\omega: (G_n(\omega) \setminus U_{\varepsilon_i} G_0(\omega)) \cap K \neq \emptyset\} < \varepsilon_i \quad \forall n \geqslant n_i.$$

Now consider the sequence $(\tilde{n}_i)_{n\in\mathbb{N}}$ with $\tilde{n}_1 := 0$ and $\tilde{n}_{i+1} \geqslant \max\{\tilde{n}_i + 1, n_{i+1}\}$ $\forall i \geqslant 1$ and set $\alpha_n := \varepsilon_i$ $\forall n$ with $\tilde{n}_i < n \leqslant \tilde{n}_{i+1}$, $i = 1, 2, \ldots$. The sequence $(\alpha_n)_{n\in\mathbb{N}}$ constructed in this way has the desired properties.

To prove (ii) one can proceed in the same way. $\square$

Semicontinuity of single-valued deterministic multifunctions reduces to continuity of the associated function. This property remains true for the random convergence in a corresponding form.

Lemma 2.4. *Let* $G_n(\omega) = \{x_n(\omega)\}$, $x_n(\omega) \in \mathbb{R}^p$ $\forall \omega \in \Omega$, $n \in \mathbb{N}$. *Then*

$$\text{(i)}\ \left(G_n \xrightarrow{l\text{-}a.s.} G_0\right) \Leftrightarrow (x_n \xrightarrow{a.s.} x_0) \Rightarrow \left(G_n \xrightarrow{u\text{-}a.s.} G_0\right),$$

$$\text{(ii)}\ \left(G_n \xrightarrow{l\text{-}prob} G_0\right) \Leftrightarrow \left(x_n \xrightarrow{prob} x_0\right) \Rightarrow \left(G_n \xrightarrow{u\text{-}prob} G_0\right),$$

where $x_n \xrightarrow{a.s.} x_0$ *and* $x_n \xrightarrow{prob} x_0$ *denote the convergence of* $(x_n)_{n\in\mathbb{N}}$ *to* x_0 *almost surely and in probability, respectively.*

(iii) *If there exists a set* $K_0 \in C^p$ *with* $P\{\omega: x_n(\omega) \in K_0\} = 1$ $\forall n \geqslant n_0$, *then* $G_n \xrightarrow{u\text{-}a.s.} G_0$ *implies* $x_n \xrightarrow{a.s.} x_0$.

(iv) *If there exists a set* $K_0 \in C^p$ *with* $\lim_{n\to\infty} P\{\omega: x_n(\omega) \in K_0\} = 1$, *then* $\left(G_n \xrightarrow{u\text{-}prob} G_0\right)$ *implies* $x_n \xrightarrow{prob} x_0$.

Proof. The proof in the "almost surely" case is straightforward and will be omitted.

Because of

$$\{\omega: [(\{x_0(\omega)\}\setminus U_\varepsilon\{x_n(\omega)\})\cup(\{x_n(\omega)\}\setminus U_\varepsilon\{x_0(\omega)\})]\cap K \neq \emptyset\}$$
$$\subset \{\omega: d(x_0(\omega), x_n(\omega)) \geqslant \varepsilon\}$$

for arbitrary $\varepsilon > 0$ and $K \in C^p$, we obtain

$$\left(x_n \xrightarrow{\text{prob}} x_0\right) \Rightarrow \left(G_n \xrightarrow{\text{prob}} G_0\right).$$

Now, suppose that $(x_n)_{n\in\mathbb{N}}$ is not convergent in probability to x_0. Then there exist $\varepsilon > 0$, $\alpha > 0$ and an infinite set $\tilde{N} \subset \mathbb{N}$ such that $P\{\omega: d(x_l(\omega), x_0(\omega)) \geqslant \varepsilon\} \geqslant \alpha$ $\forall l \in \tilde{N}$.

We consider the sequence $(\Omega_k)_{k\in\mathbb{N}}$ with $\Omega_k = \{\omega: d(x_0(\omega), 0) \leqslant k\}$. Obviously there is a k_0 such that $P(\Omega_k) \geqslant 1 - \frac{1}{2}\alpha$ $\forall k \geqslant k_0$, hence

$$P\{\omega \in \Omega_{k_0}: d(x_l(\omega), x_0(\omega)) \geqslant \varepsilon\} \geqslant \tfrac{1}{2}\alpha \quad \forall l \in \tilde{\mathbb{N}}.$$

Since

$$P\{\omega \in \Omega: [\{x_0(\omega)\setminus U_\varepsilon\{x_l(\omega)\}]\cap\{x \in \mathbb{R}^p: d(x, 0) \leqslant k_0\} \neq \emptyset\}$$
$$\geqslant P\{\omega \in \Omega_{k_0}: d(x_l(\omega), x_0(\omega)) \geqslant \varepsilon\} \geqslant \tfrac{1}{2}\alpha,$$

$(G_n)_{n\in\mathbb{N}}$ cannot be lower semiconvergent in probability to G_0.

Let $\lim_{n\to\infty} P\{\omega\colon x_n(\omega)\in K_0\} = 1$ for some $K_0 \in C^p$ be satisfied. Then the inclusion

$$\{\omega\colon d(x_0(\omega), x_n(\omega)) \geqslant \varepsilon\}$$
$$\subset \{\omega\colon x_n(\omega)\notin K_0\}\cup\{\omega\colon [\{x_n(\omega)\}\setminus U_\varepsilon\{x_0(\omega)\}]\cap K_0 \neq \emptyset\}$$

yields

$$\left(G_n \xrightarrow{\text{u-prob}} G_0\right) \Rightarrow \left(x_n \xrightarrow{\text{prob}} x_0\right). \qquad \square$$

Finally it should be emphasized that semiconvergence (almost surely or in probability) of a sequence $(G_n)_{n\in\mathbb{N}}$ to a closed-valued multifunction G_0 is equivalent to the corresponding semiconvergence of the sequence $(\mathrm{cl}\, G_n)_{n\in\mathbb{N}}$, where cl denotes the closure.

In the following we consider the convergence of random functions.

Salinetti and Wets [25,26] introduced epi-convergence almost surely or in probability of a sequence $(f_n)_{n\in\mathbb{N}}$ as convergence almost surely or in probability of the epigraph-multifunctions. Since we are especially interested in the simultaneous approximation of the objective function and the constraint set, epi-convergence (or epi-continuity) of the objective functions is not enough as pointed out in [17] for deterministic problems. What we need is an analogue to the semicontinuity of a function with respect to the variable and the parameter. We shall call this property "semicontinuous convergence" (almost surely, in probability) because of its relation to the continuous convergence of a sequence of (deterministic) functions, concerning the relationship between continuous convergence, uniform convergence and epi-convergence we refer to [11]. In order to point out the connection to the semicontinuity, as investigated and employed in [12,17] we shall introduce the semicontinuous convergence in terms of epi-limes superior and epi-limes inferior.

Let $\{h_n, n\in\mathbb{N}\cup\{0\}\}$ be a family of deterministic functions $h_n\colon\mathbb{R}^p\to\bar{\mathbb{R}}^1$ and abbreviate

$$\mathrm{EL}_* h_n(x_0) := \sup_{V\in N\{x_0\}} \liminf_{n\to\infty} \inf_{x\in V} h_n(x)$$

(epi-limes inferior at x_0) and

$$\mathrm{EL}^* h_n(x_0) := \sup_{V\in N\{x_0\}} \limsup_{n\to\infty} \inf_{x\in V} h_n(x)$$

(epi-limes superior at x_0). $N\{x_0\}$ denotes the system of neighbourhoods of x_0.

In order to compare the semicontinuous convergence with the epi-convergence, we shall also define an epi-upper semiconvergence.

Let $\{f_n, n\in\mathbb{N}\cup\{0\}\}$ be a family of $(\Sigma^p\otimes\Sigma, \bar{\Sigma}^1)$-measurable functions and X a closed subset of $\mathbb{R}^p$.

Lemma 2.5. *The events*

$$\Omega_1^n := \{\omega\colon \forall x_0\in X\colon \mathrm{EL}_* f_n(\cdot,\omega)\,(x_0) \geqslant f_0(x_0,\omega)\}$$

and

$$\Omega_2^n := \{\omega\colon \forall x_0\in X\colon \mathrm{EL}^* f_n(\cdot,\omega)\,(x_0) \leqslant f_0(x_0,\omega)\}$$

belong to Σ for all $n\in\mathbb{N}$.

Proof. Because of our assumptions the multifunctions $\omega \to \text{cl Epi} f_n(\cdot, \omega)$, where "Epi" denotes the epigraphical multifunction, are measurable (cf. [9, Theorems 3.3 and 3.4]). Furthermore, we make use of the equations

$$\text{Epi EL}_* f_n(\cdot, \omega) = \limsup_{n\to\infty} \text{Epi} f_n(\cdot, \omega)$$

and

$$\text{Epi EL}^* f_n(\cdot, \omega) = \liminf_{n\to\infty} \text{Epi} f_n(\cdot, \omega),$$

which were firstly proved by Mosco (cf. [4]).
Obviously,

$$\limsup_{n\to\infty} \text{Epi} f_n(\cdot, \omega) = \limsup_{n\to\infty} \text{cl Epi} f_n(\cdot, \omega)$$

and

$$\liminf_{n\to\infty} \text{Epi} f_n(\cdot, \omega) = \liminf_{n\to\infty} \text{cl Epi} f_n(\cdot, \omega),$$

and the measurability of the (closed-valued) multifunctions $\omega \to \text{Epi EL}_* f_n(\cdot, \omega)$ and $\omega \to \text{Epi EL}^* f_n(\cdot, \omega)$ follows by the characterization of the Kuratowski–Mosco limits in terms of unions and intersections in [25] and [9, Proposition 2.3 and Corollary 4.2].
Employing [9, Theorems 3.3 and 4.6], we obtain the measurability of the graphs of the functions

$$\omega \to \text{EL}_* f_n(\cdot, \omega) \quad \text{and} \quad \omega \to \text{EL}^* f_n(\cdot, \omega).$$

The function f_0 having a measurable graph, we can show the measurability of the multifunctions

$$\omega \to [\text{EL}_* f_n(\cdot, \omega) - f_0(\cdot, \omega)] \cap [X \times \mathbb{R}]$$

and

$$\omega \to [\text{EL}^* f_n(\cdot, \omega) - f_0(\cdot, \omega)] \cap [X \times \mathbb{R}],$$

and finally the desired statement. $\square$

Definition 2.6. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be

(i) *lower semicontinuously convergent almost surely to f_0 on X* $\left(f_n \xrightarrow[X]{\text{l-a.s.}} f_0\right)$ if

$$P\{\omega : \forall x_0 \in X : \text{EL}_* f_n(\cdot, \omega)(x_0) \geqslant f_0(x_0, \omega)\} = 1,$$

(ii) *upper semicontinuously convergent almost surely to f_0 on X* $\left(f_n \xrightarrow[X]{\text{u-a.s.}} f_0\right)$ if

$$-f_n \xrightarrow[X]{\text{l-a.s.}} -f_0,$$

(iii) *continuously convergent almost surely to* f_0 *on* X $\left(f_n \xrightarrow[X]{\text{a.s.}} f_0\right)$ if

$$\left(f_n \xrightarrow[X]{\text{l-a.s.}} f_0\right) \wedge \left(-f_n \xrightarrow[X]{\text{l-a.s.}} -f_0\right),$$

(iv) *epi-upper semiconvergent almost surely to* f_0 *on* X $\left(f_n \xrightarrow[X]{\text{epi-u-a.s.}} f_0\right)$ if

$$P\{\omega : \forall x_0 \in X : \mathrm{EL}^* f_n(\cdot,\omega)(x_0) \leqslant f_0(x_0,\omega)\} = 1.$$

Observe that

$$\{\omega : \forall x_0 \in X : \mathrm{EL}_* f_n(\cdot,\omega)(x_0) \geqslant f_0(x_0,\omega)\}$$
$$= \{\omega : \forall x_0 \in X\ \forall (x_n)_{n\in\mathbb{N}} \text{ with } x_n \to x_0 : \liminf_{n\to\infty} f_n(x_n,\omega) \geqslant f_0(x_0,\omega)\}$$

and

$$\{\omega : \forall x_0 \in X : \mathrm{EL}^* f_n(\cdot,\omega)(x_0) \leqslant f_0(x_0,\omega)\}$$
$$= \{\omega : \forall x_0 \in X\ \exists (x_n)_{n\in\mathbb{N}} \text{ with } x_n \to x_0 : \limsup_{n\to\infty} f_n(x_n,\omega) \leqslant f_0(x_0,\omega)\}.$$

We do not introduce an epi-lower semiconvergence, because it would be the same as lower semicontinuous convergence, see [17] or [12] for the deterministic case. Epi-convergence almost surely as dealt with by Salinetti and Wets [25, 26] is then defined by

$$\left(f_n \xrightarrow[\mathbb{R}^p]{\text{epi-a.s.}} f_0\right) \Leftrightarrow \left(f_n \xrightarrow[\mathbb{R}^p]{\text{l-a.s.}} f_0\right) \wedge \left(f_n \xrightarrow[\mathbb{R}^p]{\text{epi-u-a.s.}} f_0\right).$$

An equivalent characterization of the epi-convergence almost surely is given by the convergence almost surely of the epigraphical multifunctions.

Furthermore, we have the following relations, which will be useful for the derivation of an appropriate description of semiconvergence in probability.

Lemma 2.7. *Let* $X \subset \mathbb{R}^p$ *be a closed set and* $\{h_n,\ n \in \mathbb{N} \cup \{0\}\}$ *a family of deterministic functions* $h_n : \mathbb{R}^p \to \bar{\mathbb{R}}^1$. *Then*

(i) $(\forall x_0 \in X : \mathrm{EL}_* h_n(x_0) \geqslant h_0(x_0))$
$\Leftrightarrow (\forall (\alpha_n)_{n\in\mathbb{N}}$ *with* $\alpha_n \downarrow 0$:

$$\limsup_{n\to\infty} (\mathrm{Epi}\, h_n \cap [U_{\alpha_n} X \times \mathbb{R}]) \subset (\mathrm{Epi}\, h_0 \cap [X \times \mathbb{R}])).$$

(ii) $(\forall x_0 \in X : \mathrm{EL}^* h_n(x_0) \leqslant h_0(x_0))$
$\Leftrightarrow (\liminf_{n\to\infty} \mathrm{Epi}\, h_n \supset (\mathrm{Epi}\, h_0 \cap [X \times \mathbb{R}]))$.

Proof. (i) Let the left-hand side of (i) be fulfilled and consider an arbitrary sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \downarrow 0$. Suppose that there is an $(x_0, y_0) \in \limsup_{n\to\infty} (\mathrm{Epi}\, h_n \cap [U_{\alpha_n} X \times \mathbb{R}])$. Then we find a sequence $(x_n, y_n)_{n \in \tilde{N} \subset \mathbb{N}}$ with $(x_n, y_n) \to (x_0, y_0)$ and $y_n \geqslant h_n(x_n)$ $\forall n \in \tilde{N}$. Because of $\mathrm{EL}_* h_n(x_0) \geqslant h_0(x_0)$ we have

$$y_0 = \lim_{\substack{n \to \infty \\ n \in \tilde{N}}} y_n \geqslant h_0(x_0).$$

x_0 belonging to X, we obtain $(x_0, y_0) \in \operatorname{Epi} h_0 \cap [X \times \mathbb{R}]$.

Now suppose that there are an $x_0 \in X$ and a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \to x_0$ and $\liminf_{n \to \infty} h_n(x_n) < h_0(x_0) + \kappa$ for a $\kappa > 0$. To $(x_n)_{n \in \mathbb{N}}$ there is a sequence $(\alpha_n)_{n \in \mathbb{N}}$, $\alpha_n \downarrow 0$, such that $x_n \in U_{\alpha_n} X$. Hence we find $(x_0, y_0) \in \limsup_{n \to \infty} (\operatorname{Epi} h_n \cap [U_{\alpha_n} X \times \mathbb{R}])$ with $y_0 < h_0(x_0) + \kappa$. Consequently $(x_0, y_0) \notin \operatorname{Epi} h_0 \cap [X \times \mathbb{R}]$.

(ii) Let the left-hand side of (ii) be satisfied. Suppose that there is an $(x_0, y_0) \in \operatorname{Epi} h_0 \cap [X \times \mathbb{R}]$. To x_0 there is a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \to x_0$ and $\limsup_{n \to \infty} h_n(x_n) \leqslant h_0(x_0)$. To each x_n, $n \in \mathbb{N}$, we find a y_n such that $(x_n, y_n) \to (x_0, y_0)$ and $y_n \geqslant h_n(x_n)$ $\forall n \geqslant n_0$. This implies $(x_0, y_0) \in \liminf_{n \to \infty} \operatorname{Epi} h_n$.

Now suppose that there is a an $x_0 \in X$ with $\mathrm{EL}^* h_n(x_0) > h_0(x_0)$. Hence we find a neighbourhood $V \in N\{x_0\}$ such that $\limsup_{n \to \infty} \inf_{x \in V} h_n(x) > h_0(x_0)$. Consequently there is a $\kappa > 0$ with $h_n(x) > h_0(x_0) + \kappa$ for all $x \in V$ and infinitely many n. Thus to $(x_0, h_0(x_0))$ we cannot find a sequence $(x_n, y_n)_{n \in \mathbb{N}}$ with $(x_n, y_n) \to (x_0, h_0(x_0))$ and $y_n \geqslant h_n(x_n)$ $\forall n \geqslant n_0$. That means $\liminf_{n \to \infty} \operatorname{Epi} h_n \not\supseteq (\operatorname{Epi} h_0 \cap [X \times \mathbb{R}])$. $\square$

Making use of the relations (2.1)–(2.3), one also obtains

(i) $\left(h_n \xrightarrow[X]{1} h_0\right)$

$\Rightarrow (\forall \varepsilon > 0\ \forall K \in C^{p+1}\ \forall (\alpha_n)_{n \in \mathbb{N}}$ with $\alpha_n \downarrow 0\ \exists n_0\ \forall n \geqslant n_0$:

$$[(\operatorname{Epi} h_n \cap [U_{\alpha_n} X \times \mathbb{R}]) \setminus U_\varepsilon (\operatorname{Epi} h_0 \cap [X \times \mathbb{R}])] \cap K = \emptyset),$$

(ii) $\left(h_n \xrightarrow[X]{\text{epi-u}} h_0\right)$

$\Leftrightarrow (\forall \varepsilon > 0\ \forall K \in C^{p+1}\ \exists n_0\ \forall n \geqslant n_0$:

$$[(\operatorname{Epi} h_0 \cap [X \times \mathbb{R}]) \setminus U_\varepsilon (\operatorname{Epi} h_n)] \cap K = \emptyset).$$

In (i) one has an equivalence if $\operatorname{Epi} h_0$ is closed.

Semicontinuous convergence and epi-upper semiconvergence in probability can now be introduced in the following way.

Let X be a closed subset of $\mathbb{R}^p$.

Definition 2.8. The sequence $(f_n)_{n \in \mathbb{N}}$ is said to be

(i) *lower semicontinuously convergent in probability to* f_0 *on* X $\left(f_n \xrightarrow[X]{\text{l-prob}} f_0\right)$ if

$\forall \varepsilon > 0\ \forall K \in C^{p+1}\ \forall (\alpha_n)_{n \in \mathbb{N}}$ with $\alpha_n \downarrow 0$:

$$\lim_{n \to \infty} P\{\omega\colon [(\operatorname{Epi} f_n(\cdot, \omega) \cap [U_{\alpha_n} X \times \mathbb{R}]) \setminus U_\varepsilon (\operatorname{Epi} f_0(\cdot, \omega) \cap [X \times \mathbb{R}])] \cap K \neq \emptyset\} = 0,$$

(ii) *upper semicontinuously convergent in probability to* f_0 *on* X $\left(f_n \xrightarrow[X]{\text{u-prob}} f_0\right)$ if

$$-f_n \xrightarrow[X]{\text{l-prob}} -f_0,$$

(iii) *continuously convergent in probability to* f_0 *on* X $\left(f_n \xrightarrow[X]{\text{prob}} f_0\right)$ if

$$\left(f_n \xrightarrow[X]{\text{l-prob}} f_0\right) \wedge \left(f_n \xrightarrow[X]{\text{u-prob}} f_0\right),$$

(iv) *epi-upper semiconvergent in probability to* f_0 *on* X $\left(f_n \xrightarrow[X]{\text{epi-u-prob}} f_0\right)$ if

$$\forall \varepsilon > 0 \ \forall K \in C^{p+1}:$$
$$\lim_{n\to\infty} P\{\omega: [(\operatorname{Epi} f_0(\cdot,\omega)\cap[X\times\mathbb{R}])\setminus U_\varepsilon \operatorname{Epi} f_n(\cdot,\omega)]\cap K \neq \emptyset\} = 0.$$

Epi-convergence in probability is defined by

$$\left(f_n \xrightarrow[\mathbb{R}^p]{\text{epi-prob}} f_0\right) \Leftrightarrow \left(\left(f_n \xrightarrow[\mathbb{R}^p]{\text{l-prob}} f_0\right) \wedge \left(f_n \xrightarrow[\mathbb{R}^p]{\text{epi-u-prob}} f_0\right)\right).$$

Because of the $(\Sigma^p \otimes \Sigma, \bar{\Sigma}^1)$-measurability of f_n, $n \in \mathbb{N} \cup \{0\}$, all occurring events are measurable.

Concerning the relation between convergence almost surely and in probability we have the expected behaviour.

Lemma 2.9.

(i) $f_n \xrightarrow[X]{\textit{l-a.s.}} f_0$ *implies* $f_n \xrightarrow[X]{\textit{l-prob}} f_0$.

(ii) $f_n \xrightarrow[X]{\textit{epi-u-a.s.}} f_0$ *implies* $f_n \xrightarrow[X]{\textit{epi-u-prob}} f_0$.

In order to prove this lemma we cannot directly rely on the corresponding assertions for multifunctions, because we did not restrict the "convergence region" for multifunctions.

Proof of Lemma 2.9. (i) Let $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \downarrow 0$, $\varepsilon > 0$, and $K \in C^{p+1}$ be fixed. $f_n(\cdot,\omega) \xrightarrow[X]{l} f_0(\cdot,\omega)$ entails by Lemma 2.7

$$[(\operatorname{Epi} f_n(\cdot,\omega)\cap[U_{\alpha_n}X\times\mathbb{R}])\setminus U_\varepsilon(\operatorname{Epi} f_0(\cdot,\omega)\cap[X\times\mathbb{R}])]\cap K = \emptyset \quad \forall n \geqslant n_0(\omega),$$

which implies $f_n \xrightarrow[X]{\text{l-prob}} f_0$.

(ii) can be proved following a similar way. □

Finally we introduce semiconvergence for random variables. Let $\{\zeta_n, n \in \mathbb{N} \cup \{0\}\}$ be a family of random variables

$$\zeta: [\Omega, \Sigma, P] \to [\mathbb{R}^1, \Sigma^1].$$

Definition 2.10. The sequence $(\zeta_n)_{n\in\mathbb{N}}$ is said to be

(i) *lower semiconvergent almost surely to* ζ_0 $\left(\zeta_n \xrightarrow{\text{l-a.s.}} \zeta_0\right)$ if

$$P\{\omega: \liminf_{n\to\infty} \zeta_n(\omega) \geqslant \zeta_0(\omega)\} = 1,$$

(ii) *upper semiconvergent almost surely to* ζ_0 $\left(\zeta_n \xrightarrow{\text{u-a.s.}} \zeta_0\right)$ if

$$-\zeta_n \xrightarrow{\text{l-a.s.}} -\zeta_0,$$

(iii) *lower semiconvergent in probability to* ζ_0 $\left(\zeta_n \xrightarrow{\text{l-prob}} \zeta_0\right)$ if

$$\forall \varepsilon > 0: \quad \lim_{n\to\infty} P\left\{\omega: \zeta_n(\omega) < \min\left\{\zeta_0(\omega) - \varepsilon, \frac{1}{\varepsilon}\right\}\right\} = 0,$$

(iv) *upper semiconvergent in probability to* ζ_0 $\left(\zeta_n \xrightarrow{\text{u-prob}} \zeta_0\right)$ if

$$-\zeta_n \xrightarrow{\text{l-prob}} -\zeta_0.$$

3. Approximation of the constraint set

For deterministic original problems sufficient conditions on the semiconvergence in probability (with a given convergence rate) for the constraint sets are given in [31]. Here we shall show that corresponding results hold true in the more general setting of this paper.

Let $X_Q := \operatorname{cl} \bigcup_{\omega\in\Omega} Q_0(\omega)$.

Theorem 3.1. *Let the following conditions be satisfied:*

(V1) $\quad g_n^j \xrightarrow[X_Q]{\text{l-a.s.}} g_0^j, \quad j \in J,$

(V2) $\quad Q_n \xrightarrow{u\text{-}a.s.} Q_0$.

Then $\Gamma_n \xrightarrow{\text{u-a.s.}} \Gamma_0$.

The proof can be given as in the deterministic case (cf. [1, Theorem 3.1.1]). Observe that upper semiconvergence in our sense corresponds to closedness of the corresponding multifunction.

Theorem 3.2. *Let the following conditions can be satisfied*:

(V1) *The functions* $g_0^j(\cdot,\omega)$, $j \in J$, *are l.s.c. on* X_Q *for all* $\omega \in \Omega$ *and* $g_n^j \xrightarrow[X_Q]{l\text{-}prob} g_0^j$, $\forall j \in J$,

(V2) Q_0 *is closed-valued and* $Q_n \xrightarrow{u\text{-}prob} Q_0$.

Then $\Gamma_n \xrightarrow{u\text{-}prob} \Gamma_0$.

The proof makes use of the following auxiliary statements, which were already used in [31] (in a slightly different form).

Lemma 3.3. *Let* A_i, $i = 1,\dots,i_0$, *be closed sets. Then*

$$\forall \varepsilon > 0 \ \forall K \in C^p \ \exists \delta > 0 \ \forall x \in K \setminus U_\varepsilon\Big(\bigcap_{i=1}^{i_0} A_i\Big) \ \exists i \in \{1,\dots,i_0\}\colon \ x \in K \setminus U_\delta A_i.$$

For functions $h^j\colon \mathbb{R}^p \to \bar{\mathbb{R}}$ we define

$$\Gamma_h := \{x \in \mathbb{R}^p\colon h^j(x) \leqslant 0 \ \forall j \in J\}.$$

Lemma 3.4. *Let the functions* $h^j, j \in J$, *be l.s.c. on a closed set X. Then*

$$\forall K \in C^p \ \forall \delta > 0 \ \exists v > 0 \ \forall x \in (K \setminus U_\delta \Gamma_h) \cap X \ \exists j \in J\colon \quad h^j(x) > v.$$

Proof of Theorem 3.2. Assume that $(\Gamma_n)_{n\in\mathbb{N}}$ is not upper semiconvergent in probability to Γ_0. Hence there are an $\varepsilon > 0$, a $K \in C^p$, an $\alpha > 0$, and an infinite set $\tilde{N} \subset \mathbb{N}$ with the property that for

$$\Omega_k := \{\omega\colon (\Gamma_k(\omega) \setminus U_\varepsilon \Gamma_0(\omega)) \cap K \neq \emptyset\},$$

the inequalities $P(\Omega_k) > \alpha \ \forall k \in \tilde{N}$ are satisfied.

According to Lemma 3.3 to ε, K and $\omega \in \Omega$ there is a $\delta(\omega) > 0$ such that for $x \in K \setminus U_\varepsilon \Gamma_0(\omega)$ either $x \in K \setminus U_{\delta(\omega)} \tilde{\Gamma}(\omega)$ or $x \in K \setminus U_{\delta(\omega)} Q_0(\omega)$ holds. Let $A_l := \{\omega \in \Omega\colon \delta(\omega) > 1/l\}$. Then $A_l \subset A_{l+1}$ and $\bigcup_{l\in\mathbb{N}} A_l = \Omega$. Hence there is an l_1 such that $P(A_l) \geqslant 1 - \frac{1}{8}\alpha \ \forall l \geqslant l_1$. Let $\delta := 1/l_1$.

Because of Lemma 3.4 to δ there exists a $v(\omega) > 0$ with $g_0^j(x,\omega) > v(\omega)$ for all $x \in (K \setminus U_\delta \Gamma_0(\omega)) \cap X_Q$ and at least one $j \in J$. We define $J_l := \{1,\dots,l\}$, $l \in \mathbb{N}$, and $J_0(x,\omega) := \{j \in J\colon g_0^j(x,\omega) > v(\omega)\}$. Obviously $J_0(x,\omega) \neq \emptyset \ \forall x \in (K \setminus U_\delta \tilde{\Gamma}_0(\omega)) \cap X_Q$. To each $\omega \in \Omega$ and

$x_0 \in (K \setminus U_\delta \tilde{\Gamma}_0(\omega)) \cap X_Q$ we assign a $j(x_0,\omega) \in J_0(x_0,\omega)$. Since the functions $g_0^j(\cdot,\omega)$ are l.s.c. on X_Q, we find an open neighbourhood $U^\omega\{x_0\}$ such that

$$g_0^{j(x_0,\omega)}(x,\omega) > \tfrac{1}{2}\, v(\omega) \quad \forall x_0 \in X_Q \quad \forall x \in U^\omega\{x_0\}.$$

The family $\{U^\omega\{x_0\},\, x_0 \in (K \setminus U_\delta \tilde{\Gamma}_0(\omega)) \cap X_Q\}$ being an open cover of $(K \setminus U_\delta \tilde{\Gamma}_0(\omega)) \cap X_Q$, we can select a finite cover $\{U^\omega\{x_{01}\}, \ldots, U^\omega\{x_{0L}\}\}$. Hence to each $\omega \in \Omega$ there are a finite set $J_{l_0}(\omega)$ and a neighbourhood $U^\omega X_Q$ with

$$(K \setminus U_\delta \tilde{\Gamma}_0(\omega)) \cap U^\omega X_Q \subset \bigcup_{l=1}^{L} U\{x_{0l}\},$$

such that $\forall x \in (K \setminus U_\delta \tilde{\Gamma}_0(\omega)) \cap U^\omega X_Q \;\exists j \in J_{l_0}(\omega)$: $g_0^j(x,\omega) > \frac{1}{2}\, v(\omega)$. We introduce $B_l := \{\omega \in A_{l_1}$: $J_{l_0}(\omega) \subset J_l\}$ and have $B_l \subset B_{l+1}$ and $\bigcup_{l\in\mathbb{N}} B_l = A_{l_1}$. Hence there is an l_2 with $P(B_l) \geqslant 1 - \frac{1}{4}\alpha$ $\forall l \geqslant l_2$. Furthermore, let $C_l := \{\omega \in B_{l_2}$: $v(\omega) > 1/l\}$. Consequently we find an l_3 with $P(C_l) \geqslant 1 - \frac{3}{8}\alpha$ $\forall l \geqslant l_3$.

Since $g_0^j(\cdot,\omega)$ is uniformly l.s.c. on $K \cap X_Q$ for fixed ω and fixed j, to each $\omega \in C_{l_3}$ we find a $\rho(\omega) > 0$ with

$$g_0^j(x,\omega) \geqslant g_0^j(x_0,\omega) - \frac{1}{2l_3} \quad \forall x_0 \in K \cap X_Q \quad \forall x \in U_{\rho(\omega)}\{x_0\} \quad \forall j \in J_{l_2}.$$

Hence, for $D_l := \{\omega \in C_{l_3}$: $\rho(\omega) \geqslant 1/l\}$ we can choose $l_4 \geqslant \max\{2l_3, 1/\delta\}$ such that $P(D_l) \geqslant 1 - \frac{1}{2}\alpha$ $\forall l \geqslant l_4$.

Summarizing, we see that for $\omega \in D_{l_4}$ the following relation holds:

$$\forall x_0 \in (K \setminus U_\delta \tilde{\Gamma}_0(\omega)) \cap X_Q \;\; \exists j \in J_{l_2} \;\; \forall x \in U_{1/l_4}\{x_0\}: \quad g_0^j(x,\omega) \geqslant \frac{1}{l_4}. \tag{3.1}$$

Now we consider $\tilde{\Omega}_k := \Omega_k \cap D_{l_4}$. Then $P(\tilde{\Omega}_k) > \frac{1}{2}\alpha$ $\forall k \in \tilde{N}$. For $\omega \in \tilde{\Omega}_k$, $k \in \tilde{N}$, there exists an

$$x_k(\omega) \in (\tilde{\Gamma}_k(\omega) \cap Q_k(\omega) \cap K) \setminus U_\varepsilon(\tilde{\Gamma}_0(\omega) \cap Q_0(\omega)).$$

To $(Q_n)_{n\in\mathbb{N}}$ there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \downarrow 0$ such that $P\{\omega: (Q_n(\omega) \setminus U_{\alpha_n} Q_0(\omega)) \cap K \neq \emptyset\} = 0$. Hence we further distinguish the cases

(a) $x_k(\omega) \in (\Gamma_k(\omega) \cap U_{\alpha_k} X_Q \cap K) \setminus U_\varepsilon \Gamma_0(\omega)$ and

(b) $x_k(\omega) \notin U_{\alpha_k} X_Q$.

The case (b) implies $x_k(\omega) \notin U_{\alpha_k} Q_0(\omega)$, consequently

$$\omega \in \{\tilde{\omega}: (Q_k(\tilde{\omega}) \setminus U_{\alpha_k} Q_0(\tilde{\omega})) \cap K \neq \emptyset\}.$$

In case (a) we obtain either $x_k(\omega) \in (Q_k(\omega) \setminus U_\delta Q_0(\omega)) \cap K$ and hence $x_k(\omega) \notin U_{\alpha_k} Q_0(\omega)$ for $k \geqslant k_0$ or

$$x_k(\omega) \in ((\tilde{\Gamma}_k(\omega) \cap U_{\alpha_k} X_Q) \setminus U_\delta(\tilde{\Gamma}_0(\omega) \cap X_Q)) \cap K.$$

The second case entails $(x_k(\omega), 0) \in \operatorname{Epi} g_k^j(\cdot,\omega)$ $\forall j \in J$, but because of (3.1) $(x_k(\omega), 0) \notin U_{1/2l_4} \operatorname{Epi} g_0^j(\cdot,\omega)$ for at least one $j \in J_{l_2}$.

Thus, for $k \geqslant k_0$, $k \in \tilde{N}$,

$$P(\tilde{\Omega}) \leqslant P\{\omega \in \Omega\colon (Q_k(\omega) \setminus U_{\alpha_k} Q_0(\omega)) \cap K \neq \emptyset\}$$
$$+ \sum_{j \in J_{l_2}} P\{\omega \in \Omega\colon [(\operatorname{Epi} g_k^j(\cdot,\omega) \cap (U_{\alpha_k} X_Q \times \mathbb{R})) \setminus U_{1/2l_4}(\operatorname{Epi} g_0^j(\cdot,\omega) \cap (X_Q \times \mathbb{R}))]$$
$$\cap (K \times [-1, +1]) \neq \emptyset\}$$

in contradiction to the assumption. □

The proof shows that Theorem 3.2 remains true if the functions $g_0^j(\cdot,\omega)$ are l.s.c and $Q_0(\omega)$ is closed for almost all ω only.

For the following assertions on semiconvergence in probability the situation is similar.

We turn to the lower semiconvergence. Let

$$\Gamma_0^0(\omega) := \{x \in \mathbb{R}^p\colon g_0^j(x,\omega) < 0 \,\forall j \in J\},$$

$$X_\Gamma := \operatorname{cl} \bigcup_{\omega \in \Omega} \Gamma_0^0(\omega),$$

$$\Gamma_{\mathbb{R}}(\omega) := \mathbb{R}^p \setminus \Gamma_0^0(\omega),$$

and

$$CI_{\delta,K}(\omega) := \{x \in K\colon d(x, \Gamma_{\mathbb{R}}(\omega)) \geqslant \delta\} \quad (\omega \in \Omega,\ \delta > 0,\ K \in C^p).$$

Under our conditions the multifunctions Γ_0^0, $\Gamma_{\mathbb{R}}$ and $CI_{\delta,K}$ have measurable graphs.

Theorem 3.5. *Let the following assumptions be satisfied:*

(V1) *J is a finite set,*

(V2) $g_n^j \xrightarrow[X_\Gamma]{u\text{-}a.s.} g_0^j \quad \forall j \in J$,

(V3) $Q_n \xrightarrow{l\text{-}a.s.} Q_0$,

(V4) $\Gamma_0(\omega) \subset \operatorname{cl}(\Gamma_0^0(\omega) \cap Q_0(\omega)) \quad \forall \omega \in \Omega$.

Then $\Gamma_n \xrightarrow{l\text{-}a.s.} \Gamma_0$.

This theorem may be derived from [1, Theorem 3.1.5].

Theorem 3.6. *Let the following assumptions be satisfied:*

(V1) $J = \{1\}$,

(V2) $g_n^1 \xrightarrow[X_\Gamma]{epi\text{-}u\text{-}a.s.} g_0^1$, (V3) $Q_n(\omega) \equiv \mathbb{R}^p \quad \forall n \in \mathbb{N} \cup \{0\}$,

(V4) $\Gamma_0(\omega) \subset \operatorname{cl} \Gamma_0^0(\omega) \quad \forall \omega \in \Omega$.

Then $\Gamma_n \xrightarrow{l\text{-}a.s.} \Gamma_0$.

Proof. Let Ω_c be the set of all $\omega \in \Omega$ for which the functions $g_n^1(\cdot, \omega)$ epi-convergence to $g_0^1(\cdot, \omega)$ on X_Γ. Consider an $\omega \in \Omega_c$ and an $x_0 \in \Gamma_0(\omega)$. We shall show that in each neighbourhood of x_0 there is an $x \in \liminf_{n\to\infty} \Gamma_n(\omega)$. Since $\liminf_{n\to\infty} \Gamma_n(\omega)$ is closed, this entails the conclusion.

Let $U\{x_0\}$ be an arbitrary neighbourhood of x_0. Because of (V4) we find an $x \in U\{x_0\}$ with $g_0^1(x, \omega) < 0$. (V2) implies the existence of a sequence $(\hat{x}_n)_{n\in\mathbb{N}}$ with $\hat{x}_n \to x$ and $g_n^1(\hat{x}_n, \omega) \leqslant \frac{1}{2} g_0^1(x, \omega) < 0 \ \forall n \geqslant n_0$. Hence $\hat{x}_n \in \Gamma_n(\omega) \ \forall n \geqslant n_0$. Thus $x \in \liminf_{n\to\infty} \Gamma_n(\omega)$. □

Theorem 3.7. *Additionally to the assumptions* (V1) *and* (V4) *of Theorem* 3.5 *let the following conditions be satisfied*:

(V2a) *The functions* $g_0^j(\cdot, \omega)$ *are u.s.c. on* $\Gamma_0^0(\omega)$ *for all* $\omega \in \Omega$,

(V2b) $g_n^j \xrightarrow[X_\Gamma]{u\text{-}prob} g_0^j \ \forall j \in J$,

(V3) Q_0 *is closed-valued and* $Q_n \xrightarrow{l\text{-}prob} Q_0$.

Then $\Gamma_n \xrightarrow{l\text{-}prob} \Gamma_0$.

The proof makes use of the following auxiliary results (cf. [31]).

Lemma 3.8. *Let the assumptions* (V1) *and* (V4) *of Theorem* 3.5 *and the assumption* (V2a) *of Theorem* 3.7 *be satisfied. Then one has for almost all* ω

$$\forall \varepsilon > 0 \ \forall K \in C^p \ \exists \delta > 0 \ \forall x \in \Gamma_0(\omega) \cap K \ \exists x_1 \in Q_0(\omega) \cap \Gamma_0^0(\omega):$$
$$U_\delta\{x_1\} \subset U_\varepsilon\{x\} \cap \Gamma_0^0(\omega).$$

Lemma 3.9. *Let the assumptions* (V1) *of Theorem* 3.5 *and* (V2a) *of Theorem* 3.7 *be satisfied. Then one has for almost all* ω

$$\forall K \in C^p \ \forall \delta > 0 \ \exists \nu > 0 \ \forall x \in CI_{\delta,K}(\omega) \ \forall j \in J: \quad g_0^j(x, \omega) \leqslant -\nu.$$

Proof of Theorem 3.7. Suppose that there exist an $\varepsilon > 0$, a $K \in C^p$, an $\alpha > 0$, and an infinite set $\tilde{N} \subset \mathbb{N}$ such that for

$$\Omega_k := \{\omega: [\Gamma_0(\omega) \setminus U_\varepsilon \Gamma_k(\omega)] \cap K \neq \emptyset\}$$

the inequalities $P(\Omega_k) > \alpha \ \forall k \in \tilde{N}$ are satisfied.

We investigate the behaviour of g_0^j. Because of Lemma 3.8 to ω, $\frac{1}{2}\varepsilon$, and K there is a $\delta(\omega) > 0$ with the property that to each $x_0(\omega) \in \Gamma_0(\omega) \cap K$ there exists a ball $U_{\delta(\omega)}\{\hat{x}(\omega)\} \subset U_{\varepsilon/2}\{x_0(\omega)\}$ with $U_{\delta(\omega)}\{\hat{x}(\omega)\} \subset \Gamma_0^0(\omega)$ and $\hat{x}(\omega) \in Q_0(\omega)$.

Let $A_l := \{\omega \in \Omega: \delta(\omega) > 1/l\}$. Since $A_l \subset A_{l+1}$ and $\bigcup_{l \in \mathbb{N}} A_l = \Omega$, we find an $l_1 \in \mathbb{N}$ such that $P(A_l) \geqslant 1 - \frac{1}{4}\alpha \ \forall l \geqslant l_1$. The construction of A_{l_1} ensures that

$$U_{1/2l_1}\{\hat{x}(\omega)\} \subset CI_{1/2l_1,K}(\omega) \cap U_{\varepsilon/2}K$$

for $\omega \in A_{l_1}$ is valid. Lemma 3.9 guarantees the existence of a $v(\omega) > 0$ with

$$g_0^j(x, \omega) \leqslant -v(\omega) \quad \forall j \in J \ \forall x \in CI_{1/2l_1,K}(\omega).$$

Now we consider the sequence $(B_l)_{l \in \mathbb{N}}$ with $B_l := \{\omega \in A_{l_1}: v(\omega) > 1/l\}$. Because of $B_l \subset B_{l+1}$ and $\bigcup_{l \in \mathbb{N}} B_l = A_{l_1}$ we find an $l_2 \geqslant 2l_1$ such that $P(B_l) \geqslant 1 - \frac{1}{2}\alpha \ \forall l \geqslant l_2$ holds. Thus for $\omega \in B_{l_1}$ we have the following assertion:

$$\forall x_0 \in \Gamma_0(\omega) \cap K \ \ \exists \hat{x}(\omega) \in U_{\varepsilon/2}\{x_0\} \cap \Gamma_0^0(\omega) \cap Q_0(\omega):$$

$$\sup_{j \in J} \sup_{x \in U_{1/2l_1}\{\hat{x}(\omega)\}} g_0^j(x, \omega) < -\frac{1}{l_2}. \tag{3.2}$$

We return to the sets Ω_k and abbreviate $\tilde{\Omega}_k := \Omega_k \cap B_{l_2}$. Then we have $P(\tilde{\Omega}_k) > \frac{1}{2}\alpha \ \forall k \in \tilde{N}$. For $\omega \in \tilde{\Omega}_k$, $k \in \tilde{N}$, there exists an $x_k(\omega) \in \Gamma_0(\omega) \cap K$ with $x_k(\omega) \notin U_\varepsilon \Gamma_k(\omega)$. ω belonging to B_{l_2}, to $x_k(\omega)$ we find an $\hat{x}_k(\omega) \in U_{\varepsilon/2}\{x_k(\omega)\} \cap \Gamma_0^0(\omega) \cap Q_0(\omega)$ with the property (3.2), hence $(\tilde{x}_k(\omega), 0) \notin U_{1/4l_2} \mathrm{Epi}(-g_0^j(\cdot, \omega))$ for all $\tilde{x}_k(\omega) \in U_{1/4l_2}\{\hat{x}_k(\omega)\}$ and all $j \in J$. Furthermore, $U_{1/4l_2}\{\hat{x}_k(\omega)\} \subset X_\Gamma$.

Now two cases are possible: If there is an $\tilde{x}_k(\omega) \in U_{l/4l_2}\{\hat{x}_k(\omega)\}$ which does not belong to $\tilde{\Gamma}(\omega)$, there must be a $j \in J$ with $g_k^j(\tilde{x}_k(\omega), \omega) > 0$, thus $(\tilde{x}_k(\omega), 0) \in \mathrm{Epi}(-g_k^j(\cdot, \omega))$. Otherwise, if $U_{1/4l_2}\{\hat{x}_k(\omega)\} \subset \tilde{\Gamma}_k(\omega)$, we obtain $U_{l/4l_2}\{\hat{x}_k(\omega)\} \cap Q_k(\omega) = \emptyset$. Consequently, because of $\hat{x}_k(\omega) \in Q_0(\omega)$ we have $Q_0(\omega) \setminus U_{1/4l_2} Q_k(\omega) \neq \emptyset$. Summarizing,

$$\begin{aligned} P(\tilde{\Omega}_k) \leqslant\; & P\{\omega \in \Omega: Q_0(\omega) \setminus U_{1/4l_2} Q_k(\omega) \neq \emptyset\} \\ & + \sum_{j \in J} P\{\omega \in \Omega: [(\mathrm{Epi}(-g_k^j(\cdot, \omega)) \cap (X_\Gamma \times \mathbb{R}) \setminus U_{1/4l_2}(\mathrm{Epi}(-g_0^j(\cdot, \omega)) \cap (X_\Gamma \times \mathbb{R}))] \\ & \qquad \cap (K \times [-1, +1]) \neq \emptyset\}, \end{aligned}$$

which yields a contradiction. □

The assumption (V4) of Theorem 3.5 or Theorem 3.6 cannot be fulfilled for equality constraints. However, convergence results (almost surely, in probability) for equality constraints under convexity assumptions that correspond to [1, Theorem 3.2.2] can be proved in this setting too (cf. [30]).

4. Stability

We shall now investigate the behaviour of the optimal values and the solution sets. Theorems 4.1 and 4.2 and Corollary 4.3 are a.s. variants of well-known stability theorems of parametric programming (cf. [1, 17]). They can be immediately derived from the corresponding deterministic assertions.

We use the abbreviation $\hat{X} := \mathrm{cl} \bigcup_{\omega \in \Omega} \Gamma_0(\omega)$, and we shall confine the investigations to proper objective functions, i.e., functions with values in $(-\infty, +\infty]$, which are not identically $+\infty$.

Theorem 4.1. (i) $\Phi_n \xrightarrow{u\text{-}a.s.} \Phi_0$ *if the assumptions*

(V1a) $f_n \xrightarrow{u\text{-}a.s.} f_0$ *or*

(V1a′) $\exists \bar{x}_0$ *with* $P\{\omega: \bar{x}_0 \in \Psi_0(\omega)\} = 1$ *and* $f_n \xrightarrow[\{\bar{x}_0\}]{u\text{-}a.s.} f_0$, *and*

(V2a) $\Gamma_n \xrightarrow{l\text{-}a.s.} \Gamma_0$

are satisfied.

(ii) $\Phi_n \xrightarrow{l\text{-}a.s.} \Phi_0$ *if the assumptions*

(V1b) $f_n \xrightarrow[\hat{X}]{l\text{-}a.s.} f_0$,

(V2b) $\Gamma_n \xrightarrow{u\text{-}a.s.} \Gamma_0$, *and*

(V3) $\exists K \in C^p\ \exists n_0 \in \mathbb{N}\ \forall n \geqslant n_0$: $P\{\omega: \Gamma_n(\omega) \subset K\} = 1$
are fulfilled.

(iii) $\Psi_n \xrightarrow{u\text{-}a.s.} \Psi_0$ *if* $\Phi_n \xrightarrow{u\text{-}a.s.} \Phi_0$ *holds and the assumptions* (V1b) *and* (V2b) *are satisfied.*

Theorem 4.2. *Let* $\Gamma_n(\omega) = \mathbb{R}^p\ \forall \omega \in \Omega, \forall n \in \mathbb{N}$. *Then* $\Phi_n \xrightarrow{u\text{-}a.s.} \Phi_0$ *if the assumptions*

(V) $f_n \xrightarrow[\mathbb{R}^p]{epi\text{-}u\text{-}a.s.} f_0$ *or*

(V′) $\exists \bar{x}_0$ *with* $P\{\omega\colon \bar{x}_0 \in \Psi_0(\omega)\} = 1$ *and* $f_n \xrightarrow[\{\bar{x}_0\}]{epi\text{-}u\text{-}a.s.} f_0$

are fulfilled.

Corollary 4.3. *Let* $\Gamma_n(\omega) = \mathbb{R}^p \ \forall \omega \in \Omega \ \forall n \in \mathbb{N}$. *Then*

$$\left(f_n \xrightarrow[\mathbb{R}^p]{epi\text{-}a.s.} f_0\right) \Rightarrow \left(\Psi_n \xrightarrow{u\text{-}a.s.} \Psi_0\right).$$

Now we shall deal with convergence in probability. The following conditions will be used:

(VP1a) $f_0(\cdot,\omega)$ is u.s.c. on $\hat{X}$ for all $\omega \in \Omega$ and

$$f_n \xrightarrow[\hat{X}]{\text{u-prob}} f_0,$$

(VP1b) $f_0(\cdot,\omega)$ is l.s.c. on $\hat{X}$ for all $\omega \in \Omega$ and

$$f_n \xrightarrow[\hat{X}]{\text{l-prob}} f_0,$$

(VP1a′) there exists an $\bar{x}_0$ such that for all $\omega \in \Omega$ $\bar{x}_0 \in \bar{\Psi}_0(\omega)$, $f_0(\cdot,\omega)$ is u.s.c. at $\bar{x}_0$ and $f_n \xrightarrow[\{\bar{x}_0\}]{\text{u-prob}} f_0$,

(VP2a) $\Gamma_n \xrightarrow{\text{l-prob}} \Gamma_0$,

(VP2b) $\Gamma_n \xrightarrow{\text{u-prob}} \Gamma_0$,

(VP3) $\exists K \in C^p\colon \lim_{n\to\infty} P\{\omega\colon \Gamma_n(\omega) \subset K\} = 1$.

Theorem 4.4.

(i) $\Phi_n \xrightarrow{u\text{-}prob} \Phi_0$ *if* (VP1a) *or* (VP1a′) *and* (VP2a) *hold.*

(ii) $\Phi_n \xrightarrow{l\text{-}prob} \Phi_0$ *if* (VP1b), (VP2b) *and* (VP3) *are satisfied.*

(iii) $\Psi_n \xrightarrow{u\text{-}prob} \Psi_0$ *if* $\Phi_n \xrightarrow{u\text{-}prob} \Phi_0$ *and* (VP1b) *and* (VP2b) *are fulfilled.*

Proof. (i) Suppose that there exist an $\varepsilon \in (0,1)$, an $\alpha > 0$, and an infinite set $\tilde{N} \subset \mathbb{N}$ such that for

$$\Omega_k := \left\{\omega \in \Omega\colon \Phi_k(\omega) > \max\left\{\Phi_0(\omega) + \varepsilon, -\frac{1}{\varepsilon}\right\}\right\},$$

the inequalities $P(\Omega_k) > \alpha \ \forall k \in \tilde{N}$ hold.

If (VP1a) is satisfied, we choose an $x_0(\omega) \in \Gamma_0(\omega)$ with $f_0(x_0(\omega), \omega) < \max\{\Phi_0(\omega) + \frac{1}{2}\varepsilon, -2/\varepsilon\}$ and put $\tilde{X} := \hat{X}$. Otherwise we define $x_0(\omega) := \bar{x}_0 \ \forall \omega \in \Omega$ and $\tilde{X} := \{\bar{x}_0\}$.

Let K_j be the closed ball with centre 0 and radius j. We introduce the sets $A_l := \{\omega \in \Omega\colon x_0(\omega) \in K_l\}$ $l \in \mathbb{N}$.

Because of $A_l \subset A_{l+1}$ and $\bigcup_{l \in \mathbb{N}} A_l = \Omega$ there is an l_1 with $P(A_l) \geqslant 1 - \frac{1}{8}\alpha \ \forall l \geqslant l_1$. $f_0(\cdot, \omega)$ being uniformly u.s.c on $\tilde{X} \cap K_{l_1}$ for all $\omega \in \Omega$, to $\omega \in A_{l_1}$ and we find a $\delta(\omega) > 0$ such that

$$f_0(x,\omega) \leqslant f_0(x_0(\omega),\omega) + \tfrac{1}{4}\varepsilon \quad \forall x_0 \in K_{l_1} \ \forall x \in U_{\delta(\omega)}\{x_0(\omega)\}.$$

For $B_l := \{\omega \in A_{l_1}\colon \delta(\omega) > 1/l\}$ we have $B_l \subset B_{l+1}$ and $\bigcup_{l \in \mathbb{N}} B_l = A_{l_1}$, thus we can choose an l_2 in such a way that

$$P(B_l) \geqslant 1 - \tfrac{1}{4}\alpha \quad \forall l \geqslant l_2 \geqslant \frac{2}{\varepsilon}.$$

Finally, we define $D_l := \{\omega \in B_{l_2}\colon |f_0(x_0(\omega),\omega)| \leqslant l\}$ and choose $l_3 \geqslant 2l_2$ with $P(D_l) \geqslant 1 - \frac{1}{2}\alpha$ $\forall l \geqslant l_3$. Hence, for $\omega \in D_{l_3}$ we have the following relations:

$$f_0(x,\omega) \leqslant f_0(x_0(\omega),\omega) + \tfrac{1}{4}\varepsilon \quad \forall x_0 \in K_{l_1} \ \forall x \in U_{1/l_2}\{x_0(\omega)\}$$

and

$$|f_0(x_0(\omega),\omega)| \leqslant l_3.$$

Let $\hat{\varepsilon} := 1/2l_2$ and $\hat{K} := K_{l_1}$. By definition of l_2, $\hat{\varepsilon} \leqslant \frac{1}{4}\varepsilon$. According to Proposition 2.3, to $\hat{K}$ and $(\Gamma_n)_{n \in \mathbb{N}}$ there is a sequence $(\alpha_n)_{n \in \mathbb{N}}$ with $\alpha_n \downarrow 0$ and $P\{\omega\colon [\Gamma_0(\omega) \setminus U_{\alpha_n}\Gamma_n(\omega)] \cap \hat{K} \neq \emptyset\} = 0$. Obviously there is a k_0 with $\alpha_k \leqslant \frac{1}{4}\hat{\varepsilon} \ \forall k \geqslant k_0$.

Now, let $\tilde{\Omega}_k := \Omega_k \cap D_{l_3}$. Because of the assumption we have

$P(\tilde{\Omega}_k) > \frac{1}{2}\alpha \ \forall k \in \tilde{N}$. We consider $\omega \in \tilde{\Omega}_k$, $k \in \tilde{N}$, $k \geqslant k_0$.

If $\Gamma_k(\omega) = \emptyset$ we have

$$[\Gamma_0(\omega) \setminus U_{\alpha_k}\Gamma_k(\omega)] \cap \hat{K} \neq \emptyset.$$

Otherwise we can choose an $x_k(\omega) \in \Gamma_k(\omega)$ such that $d(x_k(\omega), x_0(\omega)) \leqslant \inf_{x \in \Gamma_k(\omega)} d(x, x_0(\omega)) + \frac{1}{2}\alpha_k$. Consequently, we obtain $f_k(x_k(\omega),\omega) \geqslant \Phi_k(\omega) > \max\{\Phi_0(\omega) + \varepsilon, -1/\varepsilon\} > f_0(x_0(\omega),\omega) + \frac{1}{2}\varepsilon$.

We distinguish two cases:

If $d(x_k(\omega), x_0(\omega)) \geqslant 2\alpha_k$, we obtain $[\Gamma_0(\omega) \setminus U_{\alpha_k}\Gamma_k(\omega)] \cap \hat{K} \neq \emptyset$. If $d(x_k(\omega), x_0(\omega)) < 2\alpha_k$, then $x_k(\omega)$ belongs to $U_{\hat{\varepsilon}}\{x_0(\omega)\} \cap U_{2\alpha_k}\hat{X}$ and $U_{\hat{\varepsilon}}\{x_k(\omega)\} \subset U_{2\hat{\varepsilon}}\{x_0(\omega)\}$ holds. Hence,

$$\sup_{\tilde{x} \in U_{\hat{\varepsilon}}\{x_k(\omega)\}} f_0(\tilde{x},\omega) \leqslant f_0(x_0(\omega),\omega) + \tfrac{1}{4}\varepsilon < f_k(x_k(\omega),\omega) - \tfrac{1}{4}\varepsilon.$$

Thus

$$(x_k(\omega),\ -f_0(x_k(\omega),\omega)-\hat{\varepsilon})$$
$$\in \mathrm{Epi}(-f_k(\cdot,\omega))\cap(U_{2\alpha_k}\hat{X}\times\mathbb{R})\cap(\mathrm{cl}\, U_{\hat{\varepsilon}}\hat{K}\times[-l_3-1,l_3+1]),$$

but $(x_k(\omega),\ -f_0(x_k(\omega),\omega)-\hat{\varepsilon})\notin U_{\hat{\varepsilon}}\,\mathrm{Epi}(-f_0(\cdot,\omega))$.

Summarizing, we obtain for $k\in\tilde{N}$, $k\geqslant k_0$,

$$P(\tilde{\Omega}_k)\leqslant P\{\omega\in\Omega\colon (\Gamma_0(\omega)\setminus U_{\alpha_k}\Gamma_k(\omega))\cap\hat{K}\neq\emptyset\}$$
$$+P\{\omega\in\Omega\colon (\mathrm{Epi}(-f_k(\cdot,\omega))\cap(U_{2\alpha_k}\hat{X}\times\mathbb{R}))\setminus(U_{\hat{\varepsilon}}\mathrm{Epi}(-f_0(\cdot,\omega))\cap[\tilde{X}\times\mathbb{R}])$$
$$\cap(\mathrm{cl}\, U_{\hat{\varepsilon}}\hat{K}\times[-l_3-1,l_3+1])\neq\emptyset\},$$

which yields a contradiction.

(ii) Suppose that there exist an $\varepsilon\in(0,1)$, an $\alpha>0$, and an infinite set $\tilde{N}\subset\mathbb{N}$ such that for

$$\Omega_k:=\left\{\omega\in\Omega\colon\left(\Phi_k(\omega)<\min\left\{\Phi_0(\omega)-\varepsilon,\frac{1}{\varepsilon}\right\}\right)\wedge(\Gamma_0(\omega)\neq\emptyset)\right\},$$

the relations $P(\Omega_k)>\alpha\ \forall k\in\tilde{N}$ hold.

Obviously we have for $\omega\in\Omega_k$, $k\in\tilde{N}$, the inequality $\Phi_0(\omega)>-\infty$. $f_0(\cdot,\omega)$ being uniformly l.s.c. on $\hat{X}\cap\mathrm{cl}\, U_1K$, to $\omega\in\Omega$ we find a $\delta(\omega)>0$ with the property

$$f_0(x,\omega)\geqslant f_0(x_0,\omega)-\tfrac{1}{4}\varepsilon\quad\forall x_0\in\hat{X}\cap\mathrm{cl}\, U_1K\ \ \forall x\in U_{\delta(\omega)}\{x_0\}.$$

Let $A_l:=\{\omega\in\Omega\colon\delta(\omega)>1/l\}$. Then there exists an $l_1\geqslant 2/\varepsilon$ with $P(A_l)\geqslant 1-\frac{1}{4}\alpha\ \forall l\geqslant l_1$. Furthermore, let $B_l:=\{\omega\in A_{l_1}\colon(|\Phi_0(\omega)|\leqslant l)\vee(\Phi_0(\omega)=+\infty)\}$ and choose l_2 such that

$$P(B_l)\geqslant 1-\tfrac{1}{2}\alpha\quad\forall l\geqslant l_2.$$

For $\omega\in B_{l_2}$ we have

$$f_0(x,\omega)\geqslant f_0(x_0,\omega)-\tfrac{1}{4}\varepsilon\quad\forall x_0\in\Gamma_0(\omega)\cap\mathrm{cl}\, U_1K\quad\forall x\in U_{1/l_1}\{x_0\},$$

and

$$|\Phi_0(\omega)|\leqslant l_2\quad\text{or}\quad\Phi_0(\omega)=+\infty.$$

Let $\hat{\varepsilon}:=1/2l_1$. Consequently $\hat{\varepsilon}\leqslant\frac{1}{4}\varepsilon$.

According to Proposition 2.3 to K and $(\Gamma_n)_{n\in\mathbb{N}}$ there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n\downarrow 0$ and $P\{\omega\in\Omega\colon(\Gamma_k(\omega)\setminus U_{\alpha_k}\Gamma_0(\omega))\cap K\neq\emptyset\}=0$. Let k_0 be such that for all $k\geqslant k_0$ the inequalities $\alpha_k\leqslant\frac{1}{4}\hat{\varepsilon}$ and $P\{\omega\in B_{l_1}\colon\Gamma_k(\omega)\subset K\}\geqslant 1-\frac{3}{4}\alpha$ are satisfied.

Now, consider $\tilde{\Omega}_k\colon=\Omega_k\cap B_{l_2}\cap\{\omega\in\Omega\colon\Gamma_k(\omega)\subset K\}$. Because of the assumption the inequalities

$$P(\tilde{\Omega}_k)\geqslant\tfrac{1}{4}\alpha\quad\forall k\geqslant k_0,\ k\in\tilde{N}$$

hold.

Let $\omega \in \tilde{\Omega}_k$, $k \in \tilde{N}$, $k \geqslant k_0$ be fixed. Then we choose $x_k(\omega) \in \Gamma_k(\omega)$ with

$$f_k(x_k(\omega), \omega) \leqslant \max\left\{\Phi_k(\omega) + \tfrac{1}{2}\varepsilon,\ \min\left\{-\frac{2}{\varepsilon}, \Phi_0(\omega) - \varepsilon\right\}\right\}.$$

Furthermore, we select $x_{0k}(\omega) \in \Gamma_0(\omega)$ such that

$$d(x_{0k}(\omega), x_k\omega)) \leqslant \inf_{x \in \Gamma_0(\omega)} d(x, x_k(\omega)) + \tfrac{1}{2}\alpha_k.$$

Thus

$$f_k(x_k(\omega), \omega) \leqslant \max\left\{\Phi_k(\omega) + \tfrac{1}{2}\varepsilon,\ \min\left\{-\frac{2}{\varepsilon}, \Phi_0(\omega) - \varepsilon\right\}\right\}$$
$$< \Phi_0(\omega) - \tfrac{1}{2}\varepsilon \leqslant f_0(x_{0k}(\omega), \omega) - \tfrac{1}{2}\varepsilon.$$

We distinguish two cases:

If $d(x_k(\omega), x_{0k}(\omega)) \geqslant 2\alpha_k$, we obtain $(\Gamma_k(\omega) \setminus U_{\alpha_k}\Gamma_0(\omega)) \cap K \neq \emptyset$.

If $d(x_k(\omega), x_{0k}(\omega)) < 2\alpha_k$ then $x_k(\omega)$ belongs to $U_{\hat{\varepsilon}}\{x_{0k}(\omega)\} \cap U_{2\alpha_k}\hat{X}$ and $U_{\hat{\varepsilon}}\{x_k(\omega)\} \subset U_{2\hat{\varepsilon}}\{x_{0k}(\omega)\}$.

Taking into account that $x_{0k}(\omega) \in \operatorname{cl} U_1 K$, we obtain

$$\inf_{\tilde{x} \in U_{\hat{\varepsilon}}\{x_k(\omega)\}} f_0(\tilde{x}, \omega) \geqslant f_0(x_{0k}(\omega), \omega) - \tfrac{1}{4}\varepsilon.$$

Now, let $\Phi_0(\omega) < \infty$ and

$$y_k(\omega) := \begin{cases} f_k(x_k(\omega), \omega) & \text{if } f_k(x_k(\omega), \omega) \geqslant -l_2 - 1, \\ -l_2 - 1 & \text{otherwise.} \end{cases}$$

Then $(x_k(\omega), y_k(\omega)) \in \operatorname{Epi} f_k(\cdot, \omega) \cap (U_{2\alpha_K}\hat{X} \times \mathbb{R}) \cap (\operatorname{cl} U_1 K \times [-l_2 - 1, l_2 + 1])$, but $(x_k(\omega), y_k(\omega)) \notin U_{\hat{\varepsilon}} \operatorname{Epi} f_0(\cdot, \omega)$, hence

$$[(\operatorname{Epi} f_k(\cdot, \omega) \cap (U_{2\alpha_k}\hat{X} \times \mathbb{R})) \setminus U_{\hat{\varepsilon}}(\operatorname{Epi} f_0(\cdot, \omega) \cap (\hat{X} \times \mathbb{R}))]$$
$$\cap (\operatorname{cl} U_1 K \times [-l_2 - 1, l_2 + 1]) \neq \emptyset.$$

If $\Phi_0(\omega) = \infty$ for $\omega \in \tilde{\Omega}_k$, the inequality $\Phi_k(\omega) < 1/\varepsilon$ follows. Consequently $f_k(x_k(\omega), \omega) < 1/\varepsilon + 1$ and

$$[(\operatorname{Epi} f_k(\cdot, \omega) \cap (U_{2\alpha_k}\hat{X} \times \mathbb{R})) \setminus U_{\hat{\varepsilon}}(\operatorname{Epi} f_0(\cdot, \omega) \cap (\hat{X} \times \mathbb{R}))] \cap (K \times [0, 1/\varepsilon + 1]) \neq \emptyset.$$

Finally, for given $\varepsilon > 0$ we consider the sets

$$\hat{\Omega}_k := \left\{\omega \in \Omega : \left(\Phi_k(\omega) < \min\left\{\Phi_0(\omega) - \varepsilon, \frac{1}{\varepsilon}\right\}\right) \wedge (\Gamma_0(\omega) = \emptyset)\right\},\ k \in \mathbb{N}. \text{ Then}$$

$$\hat{\Omega}_k \subset \{\omega : (\Gamma_k(\omega) \neq \emptyset) \wedge (\Gamma_0(\omega) \neq \emptyset)\}$$
$$\subset \{\omega : (\Gamma_k(\omega) \setminus U_\varepsilon \Gamma_0(\omega) \cap K \neq \emptyset) \vee (\Gamma_k(\omega) \not\subset K)\}.$$

Because of (VP2b) and (VP3) we obtain $\lim_{k\to\infty} P(\hat{\Omega}_k) = 0\ \forall\varepsilon > 0$.

(iii) We make use of the equality

$$\Psi_n(\omega) = \Gamma_n(\omega) \cap \{x \in \mathbb{R}^p\colon\ f_{\Phi,n}(x,\omega) \leqslant 0\},\ n \in \mathbb{N}\cup\{0\}.$$

In order to employ Theorem 3.1 we still have to show that

$$f_{\Phi,n} \xrightarrow[\hat{X}]{\text{l-prob}} f_{\Phi,0}.$$

Suppose that there exist an $\varepsilon > 0$, a $K \in C^{p+1}$, a sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \downarrow 0$, an $\alpha > 0$, and an infinite subset $\tilde{N} \subset \mathbb{N}$ such that for

$$\Omega_k := \{\omega \in \Omega\colon [(\mathrm{Epi} f_{\Phi,k}(\cdot,\omega) \cap (U_{\alpha_k}\hat{X} \times \mathbb{R})) \setminus U_\varepsilon(\mathrm{Epi} f_{\Phi,0}(\cdot,\omega) \cap (\hat{X} \times \mathbb{R}))] \cap K \neq \emptyset\},$$

the inequalities $P(\Omega_k) > \alpha\ \forall k \in \tilde{N}$ hold.

Obviously there is an $\hat{l}$ with $K \in \{x \in \mathbb{R}^{p+1}\colon d(0,x) \leqslant \hat{l}\}$.

Because of the lower semicontinuity of $f_0(\cdot,\omega)$ on $\hat{X}$ and $f_0(x,\omega) > -\infty\ \ \forall x \in \mathbb{R}^p\ \forall\omega \in \Omega$, we obtain $\inf_{x\in\hat{X}\cap\mathrm{Pr}\,K} f_0(x,\omega) > -\infty$. Consequently, there is a set

$$A_{l_1} := \left\{\omega \in \Omega\colon \inf_{x\in\hat{X}\cap\mathrm{Pr}\,K} f_0(x,\omega) > -\tfrac{1}{2}l_1\right\}$$

with $P(A_{l_1}) > \frac{3}{4}\alpha$.

Furthermore, $f_0(\cdot,\omega)$ being uniformly l.s.c on $\hat{X} \cap \mathrm{Pr}\,K$, to each $\omega \in A_{l_1}$ there is a $\rho(\omega)$ with $\inf_{x\in U_{\rho(\omega)}\hat{X}\cap\mathrm{Pr}\,K} f_0(x,\omega) > -l_1$. Hence we find an $l_2 \geqslant l_1$ and a set

$$B_{l_2} := \left\{\omega \in A_{l_1}\colon \inf_{x\in U_{1/l_2}\hat{X}\cap\mathrm{Pr}\,K} f_0(x,\omega) > -l_1\right\}$$

with $P(B_{l_2}) > \frac{1}{2}\alpha$. Finally, we can choose an $l_3 \geqslant \max\{l_2, \hat{l}+1\}$ such that for $C_{l_3} := \{\omega \in B_{l_2}\colon (\Phi_0(\omega) = -\infty) \vee (\Phi_0(\omega) > -l_3)\}$ the inequality $P(C_{l_3}) > \frac{1}{4}\alpha$ holds.

Now, let $\tilde{\varepsilon} := \min\{1/l_3, \frac{1}{2}\varepsilon, 1\}$, $\omega \in C_{l_3} \cap \Omega_k$, and suppose that $\Phi_k(\omega) > \max\{\Phi_0(\omega) + \frac{1}{2}\tilde{\varepsilon}, -2/\tilde{\varepsilon}\}$. Because of $\omega \in \Omega_k$ there is an $x_k(\omega) \in U_{\alpha_k}\hat{X} \cap \mathrm{Pr}\,K$ with

$$f_{\Phi,k}(x_k(\omega),\omega) < \min\left\{\inf_{x\in U_{\varepsilon/2}\{x_k(\omega)\}} f_{\Phi,0}(x,\omega) - \tfrac{1}{2}\varepsilon, \hat{l}\right\}.$$

Firstly, we consider the case $\Phi_0(\omega) > -\infty$. Then, with $\hat{\varepsilon} := \frac{1}{2}\varepsilon - \frac{1}{2}\tilde{\varepsilon}$ we obtain the inequality

$$\begin{aligned} f_k(x_k(\omega),\omega) &< \inf_{x\in U_{\varepsilon/2}\{x_k(\omega)\}} f_0(x,\omega) - \Phi_0(\omega) - \tfrac{1}{2}\varepsilon + \Phi_k(\omega) \\ &< \inf_{x\in U_{\varepsilon/2}\{x_k(\omega)\}} f_0(x,\omega) - \hat{\varepsilon}, \end{aligned}$$

which implies

$$[(\operatorname{Epi} f_k(\cdot,\omega)\cap(U_{\alpha_k}\hat{X}\times\mathbb{R}))\setminus U_{\tilde{\varepsilon}}(\operatorname{Epi} f_0(\cdot,\omega)\cap(\hat{X}\times\mathbb{R}))]\cap K\neq\emptyset.$$

Secondly, if $\Phi_0(\omega)=-\infty$, we have $f_{\Phi,0}(x,\omega)=+\infty$, $\forall x\in\mathbb{R}^p$. Hence $f_k(x_k(\omega),\omega)<\hat{l}+\Phi_k(\omega)<\hat{l}-2/\tilde{\varepsilon}\leqslant -l_3-1$. Consequently,

$$[(\operatorname{Epi} f_k(\cdot,\omega)\cap(U_{\alpha_k}\hat{X}\times\mathbb{R}))\setminus U_{\tilde{\varepsilon}}(\operatorname{Epi} f_0(\cdot,\omega)\cap(\hat{X}\times\mathbb{R}))]\cap K\neq\emptyset.$$

Summarizing,

$$C_{l_3}\cap\Omega_k\subset\left\{\omega\in\Omega:\ \Phi_k(\omega)>\max\left\{\Phi_0(\omega)+\frac{\tilde{\varepsilon}}{2},\ -\frac{2}{\tilde{\varepsilon}}\right\}\right\}$$
$$\cup\{\omega\in\Omega:\ [(\operatorname{Epi} f_k(\cdot,\omega)\cap(U_{\alpha_k}\hat{X}\times\mathbb{R}))\setminus U_{\tilde{\varepsilon}}(\operatorname{Epi} f_0(\cdot,\omega)\cap(\hat{X}\times\mathbb{R}))]\cap K\neq\emptyset\}.\qquad\square$$

5. Deterministic original problem

The definition of the lower semicontinuous convergence a.s. and especially in probability appears to be rather unwieldy. However, if the original problem is deterministic, there are simpler sufficient conditions.

In the following, we suppose that $f_0(x,\omega)=f_{0,D}(x)\ \forall\omega\in\Omega$.

Theorem 5.1. (i) *Let $f_{0,D}$ be l.s.c. on X. Then*

$$(\forall x_0\in X\ \forall\varepsilon>0\ \exists U\{x_0\}\in C^p:$$
$$P\{\omega:\ \liminf_{n\to\infty}\ \inf_{x\in U\{x_0\}} f_n(x,\omega)<f_{0,D}(x_0)-\varepsilon\}=0)\tag{5.1}$$
$$\Rightarrow\left(f_n\xrightarrow[X]{l\text{-}a.s.}f_{0,D}\right).$$

(ii) $(\forall x_0\in X\ \forall\varepsilon>0\ \exists U\{x_0\}\in C^p:$

$$\lim_{n\to\infty}P\left\{\omega:\ \inf_{x\in U\{x_0\}} f_n(x,\omega)<f_{0,D}(x_0)-\varepsilon\right\}=0)\tag{5.2}$$
$$\Rightarrow\left(f_n\xrightarrow[X]{l\text{-}prob}f_{0,D}\right).$$

Proof. (i) Let $K_j\subset\mathbb{R}^p$ be the closed ball with centre 0 and radius j,

$$\Omega_{k,j}:=\left\{\omega\in\Omega:\ \exists x_0\in X\cap K_j \text{ with } \mathrm{EL}_* f_n(\cdot,\omega)\,(x_0)<f_{0,D}(x_0)-\frac{1}{k}\right\},$$

and

$$\Omega_k:=\left\{\omega\in\Omega:\ \exists x_0\in X \text{ with } \mathrm{EL}_* f_n(\cdot,\omega)\,(x_0)<f_{0,D}(x_0)-\frac{1}{k}\right\}.$$

Obviously $\Omega_{k,j} \subset \Omega_{k,j+1}$ and $\bigcup_{j\in\mathbb{N}} \Omega_{k,j} = \Omega_k$ as well as $\Omega_k \subset \Omega_{k+1}$ and

$$\bigcup_{k\in\mathbb{N}} \Omega_k = \{\omega \in \Omega \colon \exists\, x_0 \in X \text{ with } \mathrm{EL}_* f_n(\cdot,\omega)\,(x_0) < f_{0,D}(x_0)\} =: \Omega_0.$$

We shall show that $P(\Omega_{k,j}) = 0\ \forall j \in N\ \forall k \in N$ holds. This entails $P(\Omega_k) = 0$ and finally $P(\Omega_0) = 0$.

Let $k \in \mathbb{N}$ and $j \in \mathbb{N}$ be fixed. According to (5.1) to every $x \in X$ there is a neighbourhood $U^k\{x\}$ such that

$$P\left\{\omega \colon \liminf_{n\to\infty} \inf_{x\in U^k\{x\}} f_n(x,\omega) < f_{0,D}(x) - \frac{1}{2k}\right\} = 0.$$

$f_{0,D}$ being l.s.c. on X, to every $x \in X$ we find a neighbourhood $\tilde{U}\{x\}$ with $f_{0,D}(\tilde{x}) \geqslant f_{0,D}(x) - 1/2k$ $\forall \tilde{x} \in \tilde{U}\{x\}$.

Let $U\{x\} := \operatorname{int}(U^k\{x\} \cap \tilde{U}\{x\})$. The family $\{U\{x\}, x \in X \cap K_j\}$ being an open cover of $X \cap K_j$, we can select a finite cover $\{U\{x_1\}, \ldots, U\{x_L\}\}$.

Now, suppose that $\omega \in \Omega_{k,j}$. Then there exists an $x_0(\omega) \in X \cap K_j$ with

$$\mathrm{EL}_* f_n(\cdot,\omega)\,(x_0(\omega)) < f_{0,D}(x_0(\omega)) - 1/k.$$

$x_0(\omega)$ belongs to some $U\{x_l\}$, consequently $U^k\{x_l\}$ is a neighbourhood of $x_0(\omega)$ and we have

$$\liminf_{n\to\infty} \inf_{x\in U^k\{x_l\}} f_n(x,\omega) < f_{0,D}(x_0(\omega)) - \frac{1}{k}.$$

This entails

$$\liminf_{n\to\infty} \inf_{x\in U^k\{x_l\}} f_n(x,\omega) < f_{0,D}(x_l) - \frac{1}{2k}.$$

Summarizing,

$$P(\Omega_{k,j}) \leqslant \sum_{l=1}^{L} P\left\{\omega \colon \liminf_{n\to\infty} \inf_{x\in U^k\{x_l\}} f_n(x,\omega) < f_{0,D}(x_l) - \frac{1}{2k}\right\} = 0.$$

(ii) Let now (5.2) be fulfilled and consider a fixed $\varepsilon > 0$ and $K \in C^{p+1}$. According to (5.2) to every $x \in X$ there is a neighbourhood $U\{x\} \in C^p$ with

$$\lim_{n\to\infty} P\left\{\omega \colon \inf_{\tilde{x}\in U\{x\}} f_n(\tilde{x},\omega) < f_{0,D}(x) - \varepsilon\right\} = 0.$$

We introduce $\tilde{U}\{x\} := U_{\varepsilon/2}\{x\} \cap \operatorname{int} U\{x\}$. The family $\{\tilde{U}\{x\}, x \in \operatorname{Pr} K \cap X\}$ being an open cover of $\operatorname{Pr} K \cap X$, we can select a finite cover $\{U\{x_1\}, \ldots, U\{x_L\}\}$. Furthermore, we choose a neighbourhood UX such that

$$UX \cap \operatorname{Pr} K \subset \bigcup_{l\in\{1,\ldots,L\}} U\{x_l\}.$$

Now, let a sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \downarrow 0$ be given and consider a fixed $n \in \mathbb{N}$ with $U_{\alpha_n}X \subset UX$ and an ω such that

$$M_n(\omega) := [(\operatorname{Epi} f_n(\cdot,\omega) \cap (U_{\alpha_n}X \times \mathbb{R})) \setminus U_\varepsilon(\operatorname{Epi} f_{0,D} \cap (UX \times \mathbb{R}))] \cap K \neq \emptyset.$$

Consequently, there is at least one pair $(\tilde{x}(\omega), \tilde{y}(\omega)) \in M_n(\omega)$.

Hence $\tilde{x}(\omega)$ belongs to a $\tilde{U}\{x_l\}$, $l \in \{1, \ldots, L\}$, and we have

$$\inf_{x \in U\{x_l\}} f_n(x, \omega) \leqslant f_n(\tilde{x}(\omega), \omega) \leqslant \tilde{y}(\omega) < f_{0,D}(\tilde{x}(\omega)) - \varepsilon < f_{0,D}(x_l) - \tfrac{1}{2}\varepsilon.$$

Summarizing,

$$P\{\omega\colon M_n(\omega) \neq \emptyset\} \leqslant \sum_{l=1}^{L} P\left\{\omega\colon \inf_{x \in U\{x_l\}} f_n(x, \omega) < f_{0,D}(x_l) - \tfrac{1}{2}\varepsilon\right\}.$$

With (5.2) we obtain $f_n \xrightarrow[X]{\text{l-prob}} f_{0,D}$. □

Condition (5.1) is not necessary for the lower semicontinuous convergence a.s., see the following example.

Example 5.2. Let $p = 1$, $\Omega = [0, 1]$, P the Lebesgue-measure, and $X = \{\frac{1}{2}\}$. Furthermore, we define $f_{0,D} \equiv 1$ and for all $n \in \mathbb{N}$

$$f_n(x, \omega) = f(x, \omega) = \begin{cases} 0 & \text{if } x = \omega, \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$P\{\omega\colon \sup_{V \in \mathbb{N}\{\frac{1}{2}\}} \inf_{x \in V} f(x, \omega) = 1\} = 1, \quad \text{i.e., } f_n \xrightarrow[\{\frac{1}{2}\}]{\text{l-a.s.}} f_{0,D}.$$

On the other hand, we have for $x_0 = \frac{1}{2}$, $\varepsilon = \frac{1}{2}$, and all neighbourhoods $\hat{U}\{\frac{1}{2}\} \subset [0,1]$

$$P\{\omega\colon \inf_{x \in \hat{U}\{\frac{1}{2}\}} f(x, \omega) < \tfrac{1}{2}\} = P(\hat{U}\{\tfrac{1}{2}\}).$$

The conditions (5.1) and (5.2) have the advantage to be "pointwise" conditions. Sufficient conditions for these convergence properties are given in [31]. A more general and detailed discussion of sufficient conditions for (5.1) and (5.2) in different applications which contain the cases mentioned in the introduction will be the topic of a forthcoming paper.

Finally we shall give sufficient conditions for the epi-upper semiconvergence.

Theorem 5.3.

(i)
$$\left(\forall x_0 \in X \ \forall \varepsilon > 0\colon P\left\{\omega\colon \limsup_{n \to \infty} \inf_{x \in U_\varepsilon\{x_0\}} f_n(x, \omega) > f_{0,D}(x_0) + \varepsilon\right\} = 0\right) \tag{5.3}$$
$$\Rightarrow \left(f_n \xrightarrow[X]{\text{epi-u-a.s.}} f_{0,D}\right).$$

(ii)
$$\left(\forall x_0 \in X \ \forall \varepsilon > 0\colon \lim_{n \to \infty} P\left\{\omega\colon \inf_{x \in U_\varepsilon\{x_0\}} f_n(x, \omega) > f_{0,D}(x_0) + \varepsilon\right\} = 0\right) \tag{5.4}$$
$$\Rightarrow \left(f_n \xrightarrow[X]{\text{epi-u-prob}} f_{0,D}\right).$$

Proof. (i) Let K_j be the closed ball with centre 0 and radius j and

$$\Omega_{k,l,j} := \left\{\omega \colon \exists\, x_0 \in X \cap K_j \text{ with } \limsup_{n\to\infty} \inf_{x\in U_{1/k}\{x_0\}} f_n(x,\omega) > f_{0,D}(x_0) + \frac{1}{l}\right\}.$$

Obviously $\Omega_{k,l,j} \subset \Omega_{k,l,j+1}$ and

$$\bigcup_{j\in\mathbb{N}} \Omega_{k,l,j} = \left\{\omega \in \Omega \colon \exists\, x_0 \in X \text{ with } \limsup_{n\to\infty} \inf_{x\in U_{1/k}\{x_0\}} f_n(x,\omega) > f_{0,D}(x_0) + \frac{1}{l}\right\} =: \Omega_{k,l}.$$

Furthermore, $\Omega_{k,l} \subset \Omega_{k,l+1}$ and

$$\bigcup_{l\in\mathbb{N}} \Omega_{k,l} = \left\{\omega \in \Omega \colon \exists\, x_0 \in X \text{ with } \limsup_{n\to\infty} \inf_{x\in U_{1/k}\{x_0\}} f_n(x,\omega) > f_{0,D}(x_0)\right\} =: \Omega_k.$$

Finally, $\Omega_k \subset \Omega_{k+1}$ and

$$\bigcup_{k\in\mathbb{N}} \Omega_k = \{\omega \in \Omega \colon \exists\, x_0 \in X \text{ with } \mathrm{EL}^* f_n(\cdot,\omega)(x_0) > f_{0,D}(x_0)\} =: \Omega_0.$$

We shall show that $P(\Omega_{k,l,j}) = 0$ and can conclude that $P(\Omega_0) = 0$.

Let $k \in \mathbb{N}$, $l \in \mathbb{N}$, $j \in \mathbb{N}$ be fixed and $\delta := 1/4k$ and consider a finite cover $\{U_\delta\{x_1\}, \ldots, U_\delta\{x_L\}, x_i \in \operatorname{cl} \operatorname{Dom} f_{0,D} \cap K_j,\ i = 1, \ldots, L\}$ of $\operatorname{cl}\operatorname{Dom} f_{0,D} \cap K_j$. To every x_i there is an $\hat{x}_i \in \operatorname{cl} U_\delta\{x_i\}$ with

$$f_{0,D}(\hat{x}_i) \leqslant \inf_{\tilde{x}\in U_\delta\{x_i\}} f_{0,D}(\tilde{x}) + \frac{1}{2l}.$$

Now suppose that $\omega \in \Omega_{k,l,j}$. We consider an $x_0 \in X \cap K_j$ with

$$\limsup_{n\to\infty} \inf_{x\in U_{1/k}\{x_0\}} f_n(x,\omega) > f_{0,D}(x_0) + \frac{1}{l}.$$

To x_0 there is an $i \in \{1, \ldots, L\}$ with $x_0 \in U_\delta\{x_i\}$, and we have

$$\limsup_{n\to\infty} \inf_{x\in U_\delta\{x_i\}} f_n(x,\omega) > f_{0,D}(\hat{x}_i) + \frac{1}{2l}.$$

This entails

$$P(\Omega_{k,l,j}) \leqslant \sum_{i=1}^{L} P\left\{\omega \colon \limsup_{n\to\infty} \inf_{\tilde{x}\in U_\delta\{\hat{x}_i\}} f_n(\tilde{x},\omega) > f_{0,D}(\hat{x}_i) + \frac{1}{2l}\right\} = 0.$$

(ii) Let now (5.4) be fulfilled and consider a fixed $\varepsilon > 0$ and $K \in C^{p+1}$. We put $\delta := \frac{1}{8}\varepsilon$ and choose a finite cover $\{U_\delta\{x_1\}, \ldots, U_\delta\{x_L\}, x_i \in \operatorname{cl}\operatorname{Dom} f_{0,D} \cap \operatorname{Pr} K,\ i = 1, \ldots, L\}$ of the compact set $\operatorname{cl}\operatorname{Dom} f_{0,D} \cap \operatorname{Pr} K$. ($\operatorname{Dom} f_{0,D}$ denotes the effective domain of $f_{0,D}$) Then to every x_l we select an $\hat{x}_l \in \operatorname{cl} U_\delta\{x_l\}$ with

$$f_{0,D}(\hat{x}_l) \leqslant \inf_{\tilde{x}\in U_\delta\{x_l\}} f_{0,D}(\tilde{x}) + \delta.$$

Now let $n \in \mathbb{N}$ be fixed and ω such that

$$M_n(\omega) := ((\operatorname{Epi} f_{0,D} \cap [X \times \mathbb{R}]) \setminus U_\varepsilon(\operatorname{Epi} f_n(\cdot,\omega))) \cap K \neq \emptyset.$$

Consequently there is at least one pair $(\tilde{x}(\omega), \tilde{y}(\omega)) \in M_n(\omega)$. Hence $\tilde{x}(\omega)$ belongs to a $U_\delta\{x_l\}$, $l \in \{1, \ldots, L\}$, and we have

$$f_{0,D}(\hat{x}_l) \leqslant f_{0,D}(\tilde{x}(\omega)) + \delta \leqslant \tilde{y}(\omega) + \delta < \inf_{x \in U_{\varepsilon/2}\{\tilde{x}(\omega)\}} f_n(x, \omega) - \frac{\varepsilon}{2} + \delta$$
$$< \inf_{x \in U_\delta\{x_l\}} f_n(x, \omega) - \delta.$$

Summarizing, to ε and K there is a $\delta > 0$ such that

$$P\{\omega\colon M_n(\omega) \neq \emptyset\} \leqslant \sum_{l=1}^{L} P\left\{\omega\colon = \inf_{x \in U_\delta\{\hat{x}_l\}} f_n(x, \omega) > f_{0,D}(\hat{x}_l) + \delta\right\}.$$

This completes the proof. $\square$

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